A CHARACTERIZATION OF HYPERBOLIC MANIFOLDS

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ABSTRACT. In this note we prove that a complex manifold X is Kobayashi hyperbolic if and only if the space $\text{Hol}(\Delta, X)$ of holomorphic maps of the unit disk Δ into X is relatively compact (with respect to the compact-open topology) in the space $C(\Delta, X^*)$ of continuous maps from Δ into the one-point compactification X^* of X .

0. INTRODUCTION

Let X, Y be (connected) complex manifolds; we shall denote by $\text{Hol}(X, Y)$ and $C(X, Y)$, respectively, the space of holomorphic and continuous maps from X to Y , endowed with the compact-open topology.

Let $\delta_X: X \times X \rightarrow \mathbb{R}$ be defined by

$$\delta_X(z, w) = \inf\{\omega(0, \zeta) \mid \exists \varphi \in \text{Hol}(\Delta, X) : \varphi(0) = z, \varphi(\zeta) = w\},$$

where $z, w \in X$, Δ is the unit disk in $\mathbb{C}$, and ω is the Poincaré distance on Δ . Then the *Kobayashi (pseudo)distance* k_X on X is the largest (pseudo)distance dominated by δ_X . More precisely, let us define an *analytic chain* α connecting two points $z_0, w_0 \in X$ as a sequence of pairs

$$\alpha = \{(\zeta_0, \varphi_0), \dots, (\zeta_m, \varphi_m)\},$$

where $\zeta_0, \dots, \zeta_m \in \Delta$, $\varphi_0, \dots, \varphi_m \in \text{Hol}(\Delta, X)$ are such that $\varphi_0(0) = z_0$, $\varphi_j(\zeta_j) = \varphi_{j+1}(0)$ for $j = 0, \dots, m-1$, and $\varphi_m(\zeta_m) = w_0$. The *length* $\omega(\alpha)$ of the chain α is given by

$$\omega(\alpha) = \sum_{j=0}^m \omega(0, \zeta_j).$$

Then the Kobayashi (pseudo)distance k_X between z and $w \in X$ is given by

$$k_X(z, w) = \inf\{\omega(\alpha)\},$$

where the infimum is taken with respect to all the analytic chains connecting z to w . Since X is connected, $k_X(z, w)$ is always finite, and it is clearly the largest pseudodistance dominated by δ_X .

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The complex manifold X is said to be *hyperbolic* if k_X is an actual distance (i.e., $k_X(z, w) = 0$ implies $z = w$). In this case, the Kobayashi distance induces the original manifold topology on X [B2]. There are many examples of hyperbolic manifolds; for instance, bounded domains in $\mathbb{C}^n$, hermitian manifolds with holomorphic sectional curvature bounded above by a negative constant, anything covered by (or covering) a hyperbolic manifold, and so on. For more on definitions and properties of the Kobayashi distance and hyperbolic manifolds, see [Ko2, Ko3, L, A].

The aim of this note is to prove a new characterization of hyperbolic manifolds. A few months before Kobayashi defined hyperbolic manifolds in [Ko1], Wu [W] (see also [B1]) introduced the notion of *taut manifold*. A complex manifold X is called *taut* if $\text{Hol}(\Delta, X)$ is a normal family, that is, if every sequence $\{f_\nu\} \subset \text{Hol}(\Delta, X)$ has a subsequence $\{f_{\nu_k}\}$ that is either converging (uniformly on compact subsets) or compactly divergent (i.e., for every pair of compact sets $H \subset \Delta$ and $K \subset X$ one has $f_{\nu_k}(H) \cap K = \emptyset$ eventually).

It was immediately clear that a strong relation should exist between taut and hyperbolic manifolds; indeed, it turned out that every taut manifold is hyperbolic [Ki] and that every complete hyperbolic manifold (i.e., every manifold X with k_X complete as topological distance) is taut [Ki] (actually, it was conjectured that taut were equivalent to complete hyperbolic, until Rosay's counterexample [R]).

We shall prove a characterization of hyperbolic manifolds showing explicitly this relationship. Let X be a (connected) complex manifold, and let X^* denote its one-point (or Alexandroff) compactification. It is easy to check that X is taut iff $\text{Hol}(\Delta, X) \cup \{\infty\}$ is (closed and) compact in $C(\Delta, X^*)$, where ∞ denotes both the point at infinity of X^* and the map of constant value ∞ . Then we shall prove that X is hyperbolic iff $\text{Hol}(\Delta, X)$ is relatively compact in $C(\Delta, X^*)$ (Theorem 1.3).

It would be nice to have a similar characterization for complete hyperbolic manifolds, that is, a characterization only in terms of topological (or uniform?) properties of $\text{Hol}(\Delta, X) \hookrightarrow C(\Delta, X^*)$; but as far as I know, this is an open question.

1. HYPERBOLIC MANIFOLDS

Let (X, τ) be a noncompact connected Hausdorff locally compact topological space; its *one-point* (or *Alexandroff*) *compactification* (X^*, τ^*) is the set $X \cup \{\infty\}$, where ∞ is a point not in X , endowed with the topology

$$\tau^* = \tau \cup \{(X \setminus K) \cup \{\infty\} \mid K \subset X \text{ compact}\}.$$

It is easy to check (see [K, 5.21]) that (X^*, τ^*) is a connected Hausdorff compact topological space, with X as dense subspace. Furthermore, if X is second countable, then so is X^* , which, therefore, is metrizable [K, 4.16]. In particular, if Y is another locally compact metrizable second countable space (a manifold, for instance) then $C(Y, X^*)$ endowed with the compact-open topology is still metrizable and a subset of $C(Y, X^*)$ is compact iff it is sequentially compact.

A moment's thought shows that a sequence $\{f_\nu\} \subset C(Y, X)$ is compactly divergent iff it converges, in $C(Y, X^*)$, to the constant map ∞ ; hence it is clear that a complex manifold X is taut iff $\text{Hol}(\Delta, X)$ is relatively compact in $C(\Delta, X^*)$ and its closure is $\text{Hol}(\Delta, X) \cup \{\infty\}$, as mentioned in the introduction.

The proof of the characterization of hyperbolic manifolds we are after depends on the topological Ascoli-Arzelà theorem due to Kelley and Palais [K]. We recall the definitions involved.

Let X, Y be two topological spaces. A family $\mathcal{F} \subset C(X, Y)$ is *evenly continuous* if for every $x \in X$, $y \in Y$, and every neighbourhood U of y in Y there are a neighbourhood V of x in X and a neighbourhood W of y in Y such that for every $f \in \mathcal{F}$

$$f(x) \in W \Rightarrow f(V) \subset U.$$

In other words, points that are close to each other remain close in a uniform way under the action of elements of $\mathcal{F}$; it is a topological version of equicontinuity.

Then the topological Ascoli-Arzelà theorem is

Theorem 1.1 [K, 7.21]. *Let X be a regular locally compact topological space and Y a regular Hausdorff topological space. Then a family $\mathcal{F} \subset C(X, Y)$ is relatively compact in $C(X, Y)$ iff it is evenly continuous and $\{f(x) | f \in \mathcal{F}\}$ is relatively compact in Y for all $x \in X$.*

Since holomorphic maps contract the Kobayashi distance, $\text{Hol}(\Delta, X)$ is clearly an equicontinuous family in every hyperbolic manifold X . To get even continuity in $C(\Delta, X^*)$ we need the following

Lemma 1.2. *Let (X, d) be a noncompact, connected, locally compact metric space and (Y, d') another metric space. Then an equicontinuous family $\mathcal{F} \subset C(Y, X)$ is evenly continuous as a subset of $C(Y, X^*)$, and thus as a subset of $C(Y, X)$, too.*

Proof. Fix $x_0 \in X^*$ and $y_0 \in Y$; we must show that for any neighbourhood U of x_0 there are neighbourhoods V of y_0 and W of x_0 such that $f(y_0) \in W$ implies $f(V) \subset U$ for any $f \in \mathcal{F}$.

Assume first $x_0 \in X$. Given a neighbourhood U of x_0 , let $\varepsilon > 0$ be such that $B_d(x_0, \varepsilon)$, the open d -ball of center x_0 and radius ε , is contained in U . Being $\mathcal{F}$ equicontinuous, there is $\delta > 0$ so that

$$d'(y, y_0) < \delta \Rightarrow d(f(y), f(y_0)) < \varepsilon/2$$

for all $f \in \mathcal{F}$. We claim that $V = B_{d'}(y_0, \delta)$ and $W = B_d(x_0, \varepsilon/2)$ will do the job. Indeed, $f(y_0) \in W$ means $d(x_0, f(y_0)) < \varepsilon/2$, and so

$$d(f(y), x_0) \leq d(f(y), f(y_0)) + d(f(y_0), x_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for every $y \in V$, that is, $f(V) \subset U$.

Now assume $x_0 = \infty$, and let $U = (X \setminus K) \cup \{\infty\}$ be a neighbourhood of x_0 , where $K \subset X$ is compact. Being X locally compact, we can find $\varepsilon > 0$ such that $H = \{x \in X | d(x, K) \leq \varepsilon\}$ is still compact. Now, $\mathcal{F}$ is equicontinuous; hence there is again $\delta > 0$ so that

$$d'(y, y_0) < \delta \Rightarrow d(f(y), f(y_0)) < \varepsilon/2$$

for all $f \in \mathcal{F}$. Then $V = B_{d'}(y_0, \delta)$ and $W = (X \setminus H) \cup \{\infty\}$ will work. Indeed, $f(y_0) \in W$ means $d(f(y_0), K) > \varepsilon$; so

$$d(f(y), K) \geq d(f(y_0), K) - d(f(y), f(y_0)) > \varepsilon - \varepsilon/2 > 0,$$

for every $y \in V$, that is, $f(V) \subset X \setminus K \subset U$. $\square$

And now we can prove

Theorem 1.3. *A complex manifold X is hyperbolic if and only if $\text{Hol}(\Delta, X)$ is relatively compact in $C(\Delta, X^*)$.*

Proof. Assume X hyperbolic; then k_X is a true distance inducing the manifold topology on X [B2]. Furthermore, if we consider Δ endowed with the Poincaré distance then $\text{Hol}(\Delta, X)$ is an equicontinuous family. If X is compact, k_X is complete, X is taut [Ki], and thus $\text{Hol}(\Delta, X)$ is relatively compact in $C(\Delta, X^*)$. If X is not compact, by Lemma 1.2, $\text{Hol}(\Delta, X)$ is evenly continuous in $C(\Delta, X^*)$; being X^* compact Hausdorff (and hence regular), Theorem 1.1 shows that $\text{Hol}(\Delta, X)$ is relatively compact in $C(\Delta, X^*)$.

Conversely, suppose X is not hyperbolic and take two distinct points z_0, w_0 such that $k_X(z_0, w_0) = 0$ (for the following argument, cf. [Ki]). Choose a coordinate neighbourhood U of z_0 relatively compact in X and biholomorphic to the unit ball B of $\mathbb{C}^n$, where n is the complex dimension of X , so that $w_0 \notin \bar{U}$. Furthermore, fix another neighbourhood $V \subset \subset U$ of z_0 .

We claim that for any $\nu \in \mathbb{N}$ there is $\varphi_\nu \in \text{Hol}(\Delta, X)$ such that $\varphi_\nu(0) \in \bar{V}$ but $\varphi_\nu(\Delta_{1/\nu}) \not\subset U$, where Δ_r is the disk in $\mathbb{C}$ centered in 0 of radius r . This will yield the assertion: take $\zeta_\nu \in \Delta_{1/\nu}$ such that $\varphi_\nu(\zeta_\nu) \notin U$. If $\text{Hol}(\Delta, X)$ were relatively compact in $C(\Delta, X^*)$ then $\{\varphi_\nu\}$ would have a subsequence $\{\varphi_{\nu_k}\}$ converging to $\varphi \in C(\Delta, X^*)$; but then $\{\varphi_{\nu_k}(\zeta_{\nu_k})\}$ would converge to $\varphi(0) \in \bar{V}$, which is impossible.

So we are left to prove the claim. Assume, by contradiction, that $\nu \in \mathbb{N}$ is such that $\varphi(0) \in \bar{V}$ implies $\varphi(\Delta_{1/\nu}) \subset U$ for any $\varphi \in \text{Hol}(\Delta, X)$. Choose a constant $c > 0$ such that $\omega(0, \zeta) \geq c k_{\Delta_{1/\nu}}(0, \zeta)$ for all $\zeta \in \Delta_{1/(2\nu)}$, and let $\varepsilon = c k_U(z_0, \partial V)$. Since U is biholomorphic to B , and hence hyperbolic, $\varepsilon > 0$.

Now, $k_X(z_0, w_0) = 0$ implies that we can find a sequence of points $z_0, \dots, z_m = w_0 \in X$ such that

$$\sum_{j=1}^m \delta_X(z_{j-1}, z_j) < \frac{\varepsilon}{2},$$

which in turn implies that we can find $\zeta_j \in \Delta$ and $\varphi_j \in \text{Hol}(\Delta, X)$ for $j = 1, \dots, m$ such that $\varphi_j(0) = z_{j-1}$, $\varphi_j(\zeta_j) = z_j$ and $\sum_{j=1}^m \omega(0, \zeta_j) < \varepsilon$. Let $m_0 \leq m$ be the first integer such that $\{\varphi_{m_0}(t\zeta_{m_0}) | t \in (0, 1)\} \not\subset V$. Adding enough points of the form $t\zeta_j$ with $t \in (0, 1)$ and $j = 1, \dots, m_0$, we can assume that $\zeta_j \in \Delta_{1/(2\nu)}$, $\varphi_j(\zeta_j) \in V$ for $j = 1, \dots, m_0 - 1$ and that $\zeta_{m_0} \in \Delta_{1/(2\nu)}$, $\varphi_{m_0}(\zeta_{m_0}) \in \partial V$. Then

$$\begin{aligned} \sum_{j=1}^m \omega(0, \zeta_j) &\geq \sum_{j=1}^{m_0} \omega(0, \zeta_j) \geq c \sum_{j=1}^{m_0} k_{\Delta_{1/\nu}}(0, \zeta_j) \geq c \sum_{j=1}^{m_0} k_U(\varphi_j(0), \varphi_j(\zeta_j)) \\ &\geq c k_U(z_0, \varphi_{m_0}(\zeta_{m_0})) \geq \varepsilon, \end{aligned}$$

which is a contradiction. $\square$

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