A CHARACTERIZATION OF ABELIAN GROUPS

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ABSTRACT. Let G be a group and let $k > 2$ be an integer such that $(k^3 - k) < |G|/2$ if G is finite. Suppose that the condition $|A^2| \leq k(k+1)/2$ is satisfied by every k -element subset $A \subseteq G$. Then G is abelian.

1. INTRODUCTION

The problem considered in this paper comes from a class of problems concerning the structure of groups that satisfy the small squaring property on k -sets. This notion was introduced in [BFP] and is as follows. Let G be a group and $k > 1$ an integer. For a finite subset $A \subseteq G$ set

$$A^2 = \{a_i a_j \mid a_i, a_j \in A\}.$$

We say that G has the small squaring property on k -sets (and we write $G \in DS(k)$) if $|A^2| < k^2$ for *any* subset $A \subseteq G$ of order k . Obviously, if G is abelian then $G \in DS(k)$ for any $k > 1$, and the small squaring property may be regarded as a generalization of the commutativity property.

The classification of nonabelian $DS(k)$ groups was performed for $k = 2$ in [Fr]. For $k = 3$ the classification was basically done in [BFP], and it was completed and generalized for infinite groups in [LM].

For the general k it was shown by Neumann (see [HLM]) that if $G \in DS(k)$ then G has normal subgroups M, N with $1 \leq M < N \leq G$ such that $|M|$ and $|G/N|$ are bounded above by a function of k and N/M is abelian, i.e., G is “finite by abelian by finite”. This result was used in [HLM] to show that a group $G \in DS(k)$ if and only if G is either “abelian by finite” or “finite by elementary abelian 2-group”.

However, if G is abelian then not only $G \in DS(k)$ but also

$$|A^2| \leq k(k+1)/2$$

for any subset $A \subseteq G$ of order k . In this paper we show that this “very small” squaring property actually characterizes abelian groups.

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2. THEOREM

We will prove the following

Theorem. *Let G be a group and let $k > 2$ be an integer such that $(k^3 - k) < |G|/2$ if G is finite. Suppose that the condition $|A^2| \leq k(k+1)/2$ is satisfied by every k -element subset $A \subseteq G$. Then G is abelian.*

The basic idea of the proof is contained in the following

Lemma. *If G satisfies the conditions of the theorem then*

$$(1) \quad |B^2| \leq l(l+1)/2$$

for any l -element subset $B \subseteq G$, where $2 \leq l < k$.

Proof. The proof is by induction on l . Suppose that $\exists B \subseteq G$, $|B| = k-1$ such that $|B^2| > (k-1)k/2$. Obviously, we may assume that $|B^2| \leq k(k+1)/2$. Let $B = \{b_1, \dots, b_{k-1}\}$.

Denote $K_1^B = \{x \in G \mid xB \cap B^2 \neq \emptyset\}$ and $K_2^B = \{x \in G \mid Bx \cap B^2 \neq \emptyset\}$. Since $x \in K_1^B$ iff $\exists b_i, b_j, b_k \in B$ such that $x = (b_j b_k) b_i^{-1}$, we obtain that

$$|K_1^B| \leq |B^2| \cdot |B| \leq (k(k+1)/2)(k-1) = (k^3 - k)/2.$$

Similarly, $|K_2^B| \leq (k^3 - k)/2$, so

$$(2) \quad |K_1^B \cup K_2^B| \leq k^3 - k.$$

Choose $x \in G$ such that $x \notin K_1^B \cup K_2^B$ and let $A = B \cup \{x\}$, $|A| = k$. Consider $A^2 = B^2 \cup xB \cup Bx \cup \{x^2\}$. By the choice of x we have $|B^2 \cup xB| = |B^2| + (k-1)$. Suppose that $xB \neq Bx$. Then

$$|A^2| \geq |B^2| + k > ((k-1)k/2) + k = k(k+1)/2,$$

which is a contradiction.

Thus, we have shown that

$$(3) \quad xB = Bx \quad \forall x \in G \setminus \{K_1^B \cup K_2^B\}.$$

Denote $N_G(B) = \{x \in G \mid xBx^{-1} = B\}$. Obviously, $N_G(B) \leq G$ and, in view of (2) and (3), $|G \setminus N_G(B)| \leq k^3 - k$. If G is infinite then $N_G(B) = G$, since $|G \setminus N_G(B)| < \infty$. If G is finite, then again $N_G(B) = G$, since $|G|/2 > (k^3 - k)$ by the condition of the theorem. Thus, $gBg^{-1} = B \forall g \in G$. In particular, $b_i B = B b_i$ for all $b_i \in B$, so the sets of elements of the i th row and the i th column of the multiplication table of $B \times B$ coincide for $i = 1, \dots, k-1$. Hence $|B^2| \leq (k-1)k/2$, which is a contradiction. $\square$

We proceed now to the proof of the theorem. Suppose that a group G satisfies the condition of the theorem and G is nonabelian. By the lemma, condition (1) holds for any l -element subset $B \subseteq G$, where $2 \leq l < k$. In particular, it holds for $l = 2$, which means that $G \in DS(2)$. Then, as it is shown in [Fr], $G = Q_8 \times E$, where Q_8 is the quaternion group of order 8 and E is an elementary abelian 2-group. Take $i, j, k \in Q_8$ to be the standard generators of the quaternion group and consider the 3-element set

$$B = \{(i, 1), (j, 1), (k, 1)\} \subseteq G.$$

Then $|B^2| > 6$, which is a contradiction, since, by the lemma, condition (1) must hold for any 3-element subset of G as well. Thus G has to be abelian and the theorem is proved.

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