POSITIVE HARMONIC MAJORIZATION OF THE REAL PART OF A HOLOMORPHIC FUNCTION

STEPHEN J. GARDINER

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ABSTRACT. Let U be the unit disc. This paper investigates which domains D in the complex plane have the property that $\Re f$ belongs to h^1 , or the more restrictive property that e^f belongs to the Smirnov class $\mathcal{N}^+$, for every holomorphic function $f: U \rightarrow D$.

1. INTRODUCTION

For each domain (i.e., connected open set) D in $\mathbb{C}$, let $\mathcal{H}(U, D)$ be the class of all holomorphic functions from the open unit disc U into D . As usual, let $\mathcal{N}$ be the Nevanlinna class of all holomorphic functions f on U for which

$$\sup_{0 < r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta < \infty.$$

Each function f in $\mathcal{N}$ has a nontangential limit, denoted by $f(e^{i\theta})$, at almost every boundary point $e^{i\theta}$. The Smirnov class $\mathcal{N}^+$ is the subclass of functions f in $\mathcal{N}$ for which

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \rightarrow \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta \quad (r \rightarrow 1-).$$

A discussion of the classes $\mathcal{N}$ and $\mathcal{N}^+$ can be found in Garnett [5, Chapter II].

Theorem A below is classical; see Helms [6, Theorem 8.33] for the “if” assertion and Frostman [4, §52] or Nevanlinna [8, VII, §4.2] for the converse. Theorem B was established more recently by Ahern and Cohn [1]. For an introduction to the notion of thin sets the reader is referred to [6, Chapter 10].

Theorem A. *Let D be a domain in $\mathbb{C}$. Then $f \in \mathcal{N}$ for every f in $\mathcal{H}(U, D)$ if and only if ∂D has positive logarithmic capacity.*

Theorem B. *Let D be a domain in $\mathbb{C}$. Then $f \in \mathcal{N}^+$ for every f in $\mathcal{H}(U, D)$ if and only if $\mathbb{C} \setminus D$ is nonthin at ∞ .*

In this paper we investigate which domains D have the property that $e^f \in \mathcal{N}$, or that $e^f \in \mathcal{N}^+$, for every f in $\mathcal{H}(U, D)$. We note that $e^f \in \mathcal{N}$ if

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and only if the real part of f can be written as the difference of two positive harmonic functions on U , i.e., $\operatorname{Re} f \in h^1$. Further (assuming that $e^f \in \mathcal{N}$), $e^f \in \mathcal{N}^+$ if and only if $\operatorname{Re} f$ is majorized in U by the Poisson integral of its nontangential boundary values.

We will denote the right half-plane by D_0 . It is obviously the case that if $D \subseteq D_0$ then $\operatorname{Re} f \in h^1$ for every f in $\mathcal{H}(U, D)$. The following result describes the situation for simply connected domains that contain D_0 .

Theorem 1. *Let D be a simply connected domain that contains D_0 . Then $\operatorname{Re} f \in h^1$ for every f in $\mathcal{H}(U, D)$ if and only if*

$$(1) \quad \int_{-\infty}^{\infty} \frac{\operatorname{dist}(iy, \partial D)}{1 + y^2} dy < \infty.$$

As will be seen in §2, Theorem 1 follows easily from a known result on the angular derivative problem. Now suppose that D is a simply connected domain that contains D_0 and satisfies (1). If D_1 is a domain (not necessarily simply connected) contained in D , then clearly $\operatorname{Re} f \in h^1$ (or, equivalently, $e^f \in \mathcal{N}$) for every f in $\mathcal{H}(U, D_1)$. The following result identifies which of these domains have the stronger property that $e^f \in \mathcal{N}^+$ for every $f \in \mathcal{H}(U, D_1)$.

Theorem 2. *Let D be a simply connected domain that contains D_0 and satisfies (1), and let D_1 be a domain contained in D . Then $e^f \in \mathcal{N}^+$ for every f in $\mathcal{H}(U, D_1)$ if and only if $\mathbb{R}^4 \setminus D_1^*$ is nonthin at ∞ , where*

$$D_1^* = \{(x_1, \dots, x_4) \in \mathbb{R}^4 : (x_1^2 + x_2^2 + x_3^2)^{1/2} + ix_4 \in D_1\}.$$

Here ∞ denotes the Alexandroff point for $\mathbb{R}^4$. The condition “ $\mathbb{R}^4 \setminus D_1^*$ is nonthin at ∞ ” is equivalent to “ $D_0 \setminus D_1$ is not minimally thin at ∞ with respect to D_0 ”, but the proof of Theorem 2 (see §3) does not use any results concerning minimally thin sets.

Let $\operatorname{PI}[g]$ denote the Poisson integral in U of a function g in $L^1(\partial U)$. The following is a simple consequence of Theorem 2.

Corollary. *Let D_1 be a domain contained in D_0 . Then $u = \operatorname{PI}[u|_{\partial U}]$ for every $f = u + iv$ in $\mathcal{H}(U, D_1)$ if and only if $\mathbb{R}^4 \setminus D_1^*$ is nonthin at ∞ .*

2. PROOF OF THEOREM 1

2.1. The following result is due to Oikawa [9], who formulated it in terms of an infinite strip rather than a half-plane. A closely related result had previously been given by Rodin and Warschawski [10, Theorem 2].

Theorem C. *Let D be a simply connected domain that contains D_0 . Then (1) holds if and only if there is a one-to-one conformal map g of D onto D_0 such that $g(z)/z$ has a finite nonzero limit as $|z| \rightarrow \infty$ in $\{re^{i\theta} : |\theta| < \theta_0\}$ for each θ_0 in $(0, \pi/2)$.*

The “if” part of Theorem 1 follows easily. To see this, let D be as in the statement of Theorem 1 and suppose (1) holds. Then there is a function g as in Theorem C. Let $z = x + iy$, and put $l = \lim_{x \rightarrow \infty} g(x)/x$. Clearly l is real, so $l \in (0, \infty)$. Thus $\operatorname{Re} g$ is a positive harmonic function on D whose Poisson integral representation in D_0 includes the term lx . Hence $\operatorname{Re} g$ majorizes lx on D_0 . It follows that if $f \in \mathcal{H}(U, D)$ the function $l^{-1} \operatorname{Re} g \circ f$ is a positive harmonic majorant of $\operatorname{Re} f$, and this implies that $\operatorname{Re} f \in h^1$.

3. PROOF OF THEOREM 2

Lemma A. *Let $A \subseteq D_0$. The following are equivalent:*

- 3.2. Now let D be as in the statement of Theorem 2. It follows (see §2.1) that the subharmonic function x^+ has a harmonic majorant in D . Let h denote the least harmonic majorant of x^+ in D .

Suppose that $\mathbb{R}^4 \setminus D_1^*$ (and hence also $(D_0 \setminus D_1)^*$) is nonthin at ∞ . It follows from Lemma A that if s is any positive superharmonic function on D that majorizes h on $D \setminus D_1$ so that $s(z) \geq x$ on $D_0 \setminus D_1$, then $s(z) \geq x$ on D_0 . Hence s is a superharmonic majorant of x^+ on D , and so $s \geq h$ on D . Thus the reduced function of h relative to the set $D \setminus D_1$ in D equals h itself, and so (see Doob [3, 1.VIII.10])

$$h(z) = \int_{D \cap \partial D_1} h(w) d\mu_{z, D_1}(w) = \lim_{m \rightarrow \infty} h_m(z) \quad (z \in D_1)$$

where

$$h_m(z) = \int_{D \cap \partial D_1} \min\{h(w), m\} d\mu_{z, D_1}(w) \quad (z \in D_1)$$

and $d\mu_{z, D_1}$ denotes harmonic measure for D_1 and z . Since h is a positive harmonic function on D_1 that majorizes x there, it follows that if $f \in \mathcal{H}(U, D_1)$ then $h \circ f$ is a positive harmonic function on U that majorizes the real part of f . Further, if we define $u_m = h_m \circ f + \operatorname{Re} f - h \circ f$, then each harmonic function u_m is bounded above and so is majorized on U by the Poisson integral of its (nontangential) boundary values. It follows, on letting

$m \rightarrow \infty$, that $\Re f$ is majorized in U by the Poisson integral of its boundary values. Thus $e^f \in \mathcal{N}^+$ for every f in $\mathcal{H}(U, D_1)$.

3.3. To prove the converse, let $f = u + i\nu$, where $f: U \rightarrow D_1$ is the covering map (see Ahlfors [2, Chapter 10]). If D_1 is bounded, the result is trivial. If D_1 is unbounded then

$$(2) \quad \int_0^{2\pi} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} g(f(e^{i\theta})) \frac{d\theta}{2\pi} = \int_{\partial D_1} g d\mu_{f(w), D_1} \quad (w \in U)$$

for any continuous function g on $\overline{D_1} \cup \{\infty\}$. If we put $g(z) = \min\{x^+, m/|z|\}$ in (2) and let m tend to infinity, it follows that

$$u^+(w) \leq \int_0^{2\pi} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} u^+(e^{i\theta}) \frac{d\theta}{2\pi} = \int_{\partial D_1} x^+ d\mu_{f(w), D_1}(z) \quad (w \in U)$$

in view of the hypothesis that $e^f \in \mathcal{N}^+$. Since $f(U) = D_1$, we have

$$(\Re w)^+ \leq \int_{\partial D_1} x^+ d\mu_{w, D_1}(z) \quad (w \in D_1).$$

It follows that x^+ has a quasi-bounded harmonic majorant on D_1 and hence on $D_1 \cap D_0$. Thus x is a quasi-bounded harmonic function on $D_1 \cap D_0$, and so

$$\Re w = \int_{D_0 \cap \partial D_1} x d\mu_{w, D_1 \cap D_0}(z) \quad (w \in D_1 \cap D_0).$$

Hence the reduced function of x relative to $D_0 \setminus D_1$ in D_0 equals x itself. It follows from Lemma A that $\mathbb{R}^4 \setminus D_1^*$ is nonthin at ∞ , and this completes the proof of Theorem 2.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE, DUBLIN 4, IRELAND