AN EXAMPLE OF A HOMEOMORPHISM OF THE PLANE WHERE THE SET OF ω -LIMIT POINTS IS NOT A CLOSED SET

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ABSTRACT. Using a self-map of an interval and its associated inverse limit space we construct an example of a homeomorphism of the plane h such that the set of ω -limit points of h is not a closed set.

Let X be a metric space with $f: X \rightarrow X$ continuous. For $n > 1$ let f^n denote f composed with itself n times. A subset A is said to be a global attractor for f provided that (i) $f(A) = A$; (ii) if $x \in X$ then $d(f^n(x), A) \rightarrow 0$ as $n \rightarrow \infty$; (iii) if U is open and $A \subseteq U$ then there is an open set V and a positive integer N such that $A \subset V \subset U$, and if $n > N$ then $f^n(V) \subset U$. A point y is an ω (resp. special α) limit point of a set $Y \subset X$, provided there is a sequence of positive integers $(n(i))_{i=1}^{\infty}$ and a sequence of points $(y_i)_{i=0}^{\infty}$ such that $f^{n(i)}(y_i) = y_{i+1}$ (resp. $f^{n(i)}(y_i) = y_{i-1}$), $\lim_{i \rightarrow \infty} y_i = y$, and $y_0 \in Y$. Denote the set of ω (resp. special α) limit points of Y by $\Lambda(f, Y)$ (resp. $SA(f, Y)$).

Proposition 1. *There is a homeomorphism g of the plane $\mathbb{R}^2$ satisfying the following: (i) g has a periodic point of period m if and only if m is a power of 2; (ii) the set $\Lambda(g, \mathbb{R}^2)$ is not a closed set; (iii) $\Lambda(g, \mathbb{R}^2)$ is contained in the global attractor of g^{-1} .*

Other homeomorphisms of the plane h with $\Lambda(h, \mathbb{R}^2)$ not closed have been constructed using more geometric methods.

Associated with f is the inverse limit space $(X, f) = \{(x_0, x_1, x_2, \dots) \mid x_i \in X \text{ and } f(x_{i+1}) = x_i\}$, and the induced homeomorphism $\hat{f}: (X, f) \rightarrow (X, f)$ given by $\hat{f}((x_0, x_1, x_2, \dots)) = (f(x_0), x_0, x_1, x_2, \dots)$. If X is compact then (X, f) is compact and the projection $\pi: (X, f) \rightarrow X$ given by $\pi((x_0, x_1, x_2, \dots)) = x_0$ is continuous and closed. For a closed interval I the relationship between the dynamics on (I, f) and I has been studied by Barge and Martin in [1–3]. The spaces (I, f) are examples of what Bing has called “snakelike continua,” see [5]. In [4] homeomorphisms of $\mathbb{R}^2$ are constructed that realize these spaces as their global attractors. This result is summarized in Proposition 2.

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Proposition 2. *Let $f: I \rightarrow I$ be continuous. Then there is a homeomorphism h of the plane $\mathbb{R}^2$ and an embedding q of (I, f) into $\mathbb{R}^2$ such that $q((I, f))$ is a global attractor for h and $q\hat{f}q^{-1} = h$ on $q((I, f))$.*

For $f: X \rightarrow X$ let $\Lambda^1 = \Lambda(f, X)$. For every $n > 0$, inductively define $\Lambda^n = \Lambda(f, \Lambda^{n-1})$. The attracting center Λ^∞ is defined to be the intersection of the sets Λ^n . In [7] it is shown that for self-maps f of the interval $\Lambda^\infty = SA(f, I)$.

Proposition 3. *Let $f: X \rightarrow X$ be continuous where X is a compact metric space, h a homeomorphism of a metric space Y , and q an embedding of (X, f) into Y such that $q((X, f))$ is a global attractor for h and $q\hat{f}q^{-1} = h$ on $q((X, f))$. Then (i) $\pi(\Lambda(\hat{f}^{-1}, (X, f))) = SA(f, X)$; (ii) $\pi(\Lambda(\hat{f}, (X, f))) = \Lambda(f, X)$; (iii) $\Lambda(h^{-1}, Y) = q(\Lambda(\hat{f}^{-1}, (X, f)))$. Moreover, if $X = I$ then $\pi(\Lambda(\hat{f}^{-1}, (I, f))) = \Lambda^\infty$.*

The proof of this proposition follows more or less from the definitions. Relation (i) in Proposition 3 was pointed out to me by M. Misiurewicz during the Spring Topology Conference in Sacramento, California.

Proof of Proposition 1. Let f be a self-map of the interval such that (a) f has a periodic point of period m if and only if m is a power of 2, and (b) the set of recurrent points R is not a closed set. Such a map is constructed in [6]. In [8] it is shown that for such maps $\Lambda^\infty = R$. Let g be the inverse of the homeomorphism h given in Proposition 1. Since π is a closed map and $\pi(\Lambda(\hat{f}^{-1}, (I, f))) = \Lambda^\infty$, we have that $\Lambda(\hat{f}^{-1}, (I, f))$ is not closed. Since q is an embedding, $\Lambda(g, \mathbb{R}^2) = \Lambda(h^{-1}, \mathbb{R}^2) = q(\Lambda(\hat{f}^{-1}, (I, f)))$ is not a closed set. Now (i) follows from property (a) of f and (iii) follows from the fact that $q(\Lambda(\hat{f}^{-1}, (I, f))) \subset q((I, f))$. $\square$

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