LOWER BOUNDS FOR THE L.-S. CATEGORY OF PRODUCTS

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ABSTRACT. Halperin and Lemaire introduced L.-S. category type invariants $\text{lMcat}(A)$ and $\text{rMcat}(A)$ for certain differential algebras (A, d) . In particular, they proved that if $(A, d) = C^*(S, k)$ is the k -valued singular cochains on 1-connected space S , then these invariants are lower bounds for the classical category $\text{cat}(S)$. We use an explicit model for Ganea's space due to Felix, Halperin, Lemaire, and Thomas to prove $\text{lMcat}(A \otimes B) \leq \text{lMcat}(A) + e(B)$, over any field, where e denotes Toomer's invariant. This proves Ganea's conjecture for Mcat over fields of arbitrary characteristic.

1. INTRODUCTION

Recently, Halperin and Lemaire introduced a hierarchy of homotopy invariants (lMcat , rMcat , Acat) for certain differential graded algebras over a field k and examples of Idrissi [I] have shown that these are all distinct. For the case of singular k -valued cochains on a simply connected space S , they are all lower bounds for the Lusternik-Schnirelmann category, $\text{cat}(S)$, and can depend on the chosen field, though only on its characteristic [N].

The classical category (which is the least number less one of open subsets, contractible in S , needed to cover S) is usually difficult to compute, and lower bounds such as Toomer's invariant, $e_k(S)$, are particularly interesting and useful. It is known, for example, that e_k is additive on products [T]. Some efforts to find lower bounds for the aforementioned approximations to $\text{cat}(S)$, particularly on product spaces, have met with some limited success [J, N], and these helped in the proof of Ganea's conjecture for rational spaces [H]. Here we use similar techniques to prove

Theorem. *If (A, d) and (B, d) are differential graded cochain algebras over k with 1-connected cohomology of finite type, then*

$$\text{lMcat}_k(A) + \text{lMcat}_k(B) \geq \text{lMcat}_k(A \otimes B) \geq \text{lMcat}_k(A) + e_k(B)$$

and an identical inequality is also satisfied by rMcat_k .

For the special case $(A, d) = C^*(S, k)$ we prove

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Corollary 1. *If S and T are simply connected spaces with the homotopy type of CW-complexes and $H^*(S, k)$ and $H^*(T, k)$ are of finite k -type, then*

$$\mathrm{Mcat}_k(S) + \mathrm{Mcat}_k(T) \geq \mathrm{Mcat}_k(S \times T) \geq \mathrm{Mcat}_k(S) + e_k(T).$$

In particular,

Corollary 2. *Ganea's conjecture is true for Mcat_k , i.e., if T is a simply connected finite CW-complex, then*

$$\mathrm{Mcat}_k(T \times S^n) = \mathrm{Mcat}_k(T) + 1.$$

We remark that e_k and Mcat_k depend only on the characteristic of k [N].

2. NOTATION AND KNOWN RESULTS

We will use the notation and conventions of [HL] throughout. All algebras, vector spaces, and their tensor products are over a field k of arbitrary characteristic. Any DGA (C, d) (differential graded algebra) whose homology has finite type and is 1-connected has a model (A, d) in $k\text{-DGA}_1^*$, the cofibration category of connected, 1-connected associative differential graded cochain algebras of finite type. That is, there is a map $\varphi: (A, d) \xrightarrow{\cong} (C, d)$, which induces an isomorphism on cohomology (hereafter, such a map will be called a *quism*). There is a notion of homotopy in $k\text{-DGA}_1^*$, and the invariants we describe are invariants of homotopy type; in particular, they are preserved by quisms.

Any (A, d) in $k\text{-DGA}_1^*$ has a *free tensor model*, i.e., there is a quism of DGAs $(T(V), d) \xrightarrow{\cong} (A, d)$ where $T(V)$ is the tensor algebra on a graded vector space V over k of finite type: $V = \bigoplus V^j$, $j \geq 2$. We may also suppose that the model is *minimal*, i.e., the differential $d: V \rightarrow T^{\geq 2}(V)$, where as usual $T^p(V) = V^{\otimes p} = V \otimes \cdots \otimes V$ (p times). Morphisms in $k\text{-DGA}_1^*$ have free minimal models also. We can factor $A \xrightarrow{\varphi} B$ as $A \xrightarrow{i} A \amalg T(V) \xrightarrow{j} B$, where $A \xrightarrow{i} A \amalg T(V)$ is a minimal free extension and $\simeq$ denotes a quism.

We now define the homotopy invariants IMcat_k , rMcat_k , Acat_k , and e_k . Let $(T(V), d) \rightarrow (A, d)$ be a free minimal model for (A, d) , and suppose that $(T(V), d) \xrightarrow{j} (T(V \oplus W_m), d) \xrightarrow{\cong} (T(V)/T^{>m}(V), \bar{d})$ is a free model for the projection $T(V) \xrightarrow{\pi} T(V)/T^{>m}(V)$. We call $T(V)/T^m(V)$ the m th Ganea space of A . Call set maps $r: (T(V \oplus W_m), d) \rightarrow (T(V), d)$, which commute with the differentials and which satisfy $r \circ j = \mathrm{id}_{T(V)}$ *retracts*. Then

- (1) $\mathrm{Acat}_k(A)$ is the least m for which there is a retract, which is a morphism of graded algebras.
- (2) $\mathrm{IMcat}_k(A)$ (resp. $\mathrm{rMcat}_k(A)$) is the least m for which there is a retract, which is a morphism of graded left- $T(V)$ modules (resp. right- $T(V)$ modules).
- (3) $e_k(A)$ is the least m for which there is a retract, which is a morphism of linear spaces.

Note that Toomers's invariant $e_k(A)$ is also the largest m for which some nontrivial class in $H^*(T(V)) \cong H^*(A)$ is represented by a cycle in $T^{\geq m}(V)$. Clearly, $e_k(A) \leq \min\{\mathrm{IMcat}_k(A), \mathrm{rMcat}_k(A)\} \leq \max\{\mathrm{IMcat}_k(A), \mathrm{rMcat}_k(A)\} \leq \mathrm{Acat}_k(A)$.

In the case when $A = C^*(S)$, Halperin and Lemaire observed that $A \cong A^{\mathrm{opp}}$ and so $\mathrm{IMcat}_k(A) = \mathrm{rMcat}_k(A) \equiv \mathrm{Mcat}_k(S)$. If S is simply connected, they

also proved that $\text{cat}(S) \geq \text{Acat}_k(S)$, showing that $\text{Mcat}_k(S)$ is also a lower bound for the classical category of a space.

The idea of our proof of the theorem stated above is similar to that in [J] but we can be explicit because of an efficient model for $(T(V)/T^{>m}(V), \bar{d})$ given in [FHLT]. We describe that model here only for use in computations for IMcat ; mirror image constructions yield the required models for rMcat . (See, e.g., [N, I].)

Recall that $T(V)$ has a bigrading $T(V)^{p,q} = [T^p(V)]^{p+q}$ where $p+q$ is the total (topological) degree. This bigrading of $T(V)$ above can be carried to $T(V)/T^{>m}(V)$ by making π homogeneous of bidegree $(0, 0)$. Let $M = M^{m,*} = s(T^{m+1}(V))$, i.e., $M^{m,q} = s[T(V)]^{m+1,q}$. There is an isomorphism $\omega: T(V) \otimes M \xrightarrow{\cong} T^{>m}(V)$ defined by $\omega(\Phi \otimes s\psi) = (-1)^{|\Phi|} \Phi \cdot \psi$ where $|\Phi|$ is the topological degree of Φ . We can then define a differential D on $T(V) \otimes (k \oplus M)$ by $D = d$ on $T(V) \otimes k$ and $D = \omega - \omega^{-1} d\omega$ on $T(V) \otimes M$. This makes $(T(V) \otimes (k \oplus M), D)$ into a differential left $T(V)$ -module and an easy calculation shows that the map $(T(V) \otimes (k \oplus M), D) \xrightarrow{\psi} (T(V)/T^{>m}(V), \bar{d})$, defined by $\psi(v) = \pi(v)$ and $\psi(T(V) \otimes M) = 0$, is a quism. Using [HL, Proposition A1(i)], we can lift the quism $(T(V \oplus W_m), d) \xrightarrow{\cong} (T(V)/T^{>m}(V), \bar{d})$ over Ψ to a quism $(T(V \oplus W_m), \bar{d}) \rightarrow (T(V) \otimes (k \oplus M), D)$. Thus any retract of our new model is one of the free model and we can use our new model to compute IMcat .

Let $(T(U), d)$ and $(T(V), d)$ be minimal free tensor models for A and B respectively, and let $(T(U \oplus V), d) = (T(U), d) \amalg (T(V), d)$ be the coproduct in the category of DGAs. We construct a free model for the tensor product as follows. (See Ndombol [N] for an alternate construction, which provides an explicit formula for the differential.) Define a DGA morphism $\phi': (T(U \oplus V), d) \rightarrow (T(U), d) \otimes (T(V), d)$ by $\phi'u = u \otimes 1$ and $\phi'v = 1 \otimes v$. Thus ϕ preserves the obvious inclusions of $T(U)$ and $T(V)$. Now let

$$(T(U \oplus V), d) \rightarrow (T(U \oplus V \oplus W), d) \xrightarrow{\phi} ((T(U), d) \otimes (T(V), d))$$

be a minimal free model for ϕ' . We can thus use $(T(U \oplus V \oplus W), d)$ to compute $\text{IMcat}(A \otimes B)$, since if $T(U) \xrightarrow{\phi_A} A$ and $T(V) \xrightarrow{\phi_B} B$ are minimal free models for A and B in ADG_1^* , then $(\phi_A \otimes \phi_B) \circ \phi$ exhibits $(T(U \oplus V \oplus W), d)$ as a minimal free tensor model for $(A \otimes B, d)$.

We remark that Corollary 1 follows from the theorem and the fact that $C^*(S \times T) \sim C^*(S) \otimes C^*(T)$.

3. PROOF OF THE THEOREM

The first inequality of the theorem is proven in a manner similar to that for Mcat or cat [FH, J]. We refer the reader to [N] for details. We show here that $\text{IMcat}(A \otimes B) - e(B) \geq \text{IMcat}(A)$. The argument is essentially the second oldest in the book. Beginning in the m th Ganea space of A , we “multiply” by a longest representative α of a cohomology class in B (this is the map β) and then use the retract guaranteed by $\text{IMcat}(A \otimes B) \leq m + n$ and ϕ to get back to $A \otimes B$. Since $[\alpha] \neq 0$, we can “divide” by α and return to A . The composition of these maps is the required retract.

So assume that $e(B) = n$ and $\text{IMcat}(A \otimes B) = m + n$. (Since $\text{IMcat}(A \otimes B) \geq e(A \otimes B) = e(A) + e(B)$, $m \geq 0$.) Let X , M , and N denote $U \oplus V \oplus W$,

$sT^{m+1}(U)$, and $sT^{m+n+1}(X)$ respectively, and let

$$\begin{aligned}(T(U) \otimes (k \oplus M), D_U) &\rightarrow (T(U)/T^{>m}(U), \bar{d}), \\ (T(X) \otimes (k \oplus N), D_X) &\rightarrow (T(X)/T^{>m+n}(X), \bar{d})\end{aligned}$$

be the models for the projections as described in §2. Let

$$R: (T(X) \otimes (k \oplus N), D_X) \rightarrow (T(X), d_X)$$

be the left- $T(X)$ module map guaranteed by $\text{IMcat}(A \otimes B) = m+n$, and assume that $\alpha \in T^{\geq n}(V)$ satisfies $d\alpha = 0$ and $[\alpha] \neq 0$.

Now define $\beta: T(U) \otimes (k \oplus M) \rightarrow T(X) \otimes (k \oplus N)$ by

$$\beta(a+b) = a \otimes \alpha + \omega_x^{-1}(\omega_U(b) \otimes \alpha) \quad \text{for } a \in T(U) \text{ and } b \in T(U) \otimes M.$$

We use the fact that $D\omega^{-1} = \text{id} - \omega^{-1}d$ to show that β commutes with the differentials: First, $D\beta(a+b) = da \otimes \alpha + \omega_U(b) \otimes \alpha - \omega_x^{-1}(d\omega_U(b) \otimes \alpha)$. On the other hand, $\beta(da + Db) = da \otimes \alpha + \beta(Db)$; but $Db = D\omega_U^{-1}\omega_U(b) = \omega_U(b) - \omega_U^{-1}d\omega_U(b)$, so $D\beta = \beta D$ is clear.

Choose a graded subspace $E \subset T(V)$ containing $d(T(V))$ such that $k\langle\alpha\rangle \oplus E = T(V)$. Use this to define a left- $T(U)$ module map $\Gamma: T(U) \otimes T(V) \rightarrow T(U)$ so that $\Gamma(1 \otimes \alpha) = 1$.

We can now define a left- $T(U)$ module retract

$$r: (T(U) \otimes (k \oplus M), D) \rightarrow (T(U), d)$$

by $r = \Gamma \circ \phi \circ R \circ \beta$. It follows that $\text{IMcat}(A) \leq m$, and this completes the proof of the theorem.

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