UNIQUENESS OF TOPOLOGY FOR SEMISIMPLE LFQ -ALGEBRAS

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Dedicated to Professor Dr. G. Maltese on the occasion of his sixtieth birthday

ABSTRACT. Various classes of semisimple lmc algebras, among them barrelled Pták Q -algebras, LFQ -algebras, and FQ -algebras, have a uniquely determined topology.

INTRODUCTION

Recently Ransford [14] presented a short proof of Johnson's (uniqueness of norm) theorem [7], based on ideas of Aupetit [0]. He employs only standard properties of the spectral radius, so his argument differs from the previous ones in [0, 7].

In this paper, applying the technique of Ransford [14], we give an extension of Johnson's theorem to diverse classes of semisimple lmc algebras, among them barrelled Pták Q -algebras and LFQ -algebras (cf. Theorem 2.7 and Corollary 2.8). As a result, the algebra $O(K)$ of holomorphic functions on a suitable compact space K , as well as the algebra $\mathcal{D}(X)$ of test functions on an appropriate C^∞ -manifold X , have a uniquely determined topology (cf. Propositions 3.1 and 3.3). Furthermore, a compatible involution on a semisimple LFQ -algebra (Theorem 3.4) and any involution on a semisimple barrelled Pták Q -algebra is automatically continuous.

Uniqueness of the topology for semisimple FQ -algebras can also be derived from [0, Remark 3]. Analogous results for the commutative case can be found in [13, §14; 3]. Furthermore, questions related to the uniqueness of topology in the frame of nonnormed topological algebras are discussed in [5, 10].

1. PRELIMINARIES

In the sequel we deal with complex algebras and Hausdorff topological spaces.

An algebra E is called *semisimple* whenever its Jacobson radical J_E is trivial [1, p. 124]. An *lmc* (locally m -convex) *algebra* is a topological algebra E whose topology is defined by a family Γ_E of submultiplicative seminorms [11,

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13]. By an *F-algebra* we mean an lmc algebra whose underlying locally convex space is a *Fréchet* space. Analogous definitions hold for a *barrelled* or *Pták algebra* (for the respective locally convex space definitions, see [6, 8]). Furthermore, a *Q-algebra* will be an lmc algebra such that the set of its quasi-invertible elements is open [11, 13]. An *F-algebra* that is *Q* will be called an *FQ-algebra*. Moreover, every *Q-algebra* is *advertibly complete* [11, Theorem I, 6.4], in the sense of [16, §3] (see also [11, p. 45]). The converse is not in general true [16, p. 8].

Now, an algebra E will be called an *LF-algebra* (cf. also [11, Definition VIII, 9.1; 8, p. 223, 5]), whenever there exists an increasing sequence (E_n) , $n \in \mathbb{N}$, of *F-algebras* (subalgebras of E) such that:

- (1) $E = \bigcup_{n \in \mathbb{N}} E_n$ and each E_n is a 2-sided ideal in E ;
- (2) $\tau_n = \tau_{n+1}|_{E_n}$, where τ_n denotes the topology of E_n , $n \in \mathbb{N}$;
- (3) If τ is the final lmc topology on E induced by the natural injections $i_n: E_n \rightarrow E$, $n \in \mathbb{N}$, then $\tau|_{E_n} = \tau_n$, for every $n \in \mathbb{N}$.

An *LF-algebra* that is also *Q* will be called an *LFQ-algebra* (cf. [11, p. 301]).

Given an lmc algebra E , denote by (E_p) , $p \in \Gamma_E$, the inverse system of Banach algebras corresponding to E [11, 13]; that is, E_p is the completion of the normed algebra E/N_p , $N_p \equiv \ker(p)$, $p \in \Gamma_E$, under the norm $\|\cdot\|_p$ induced by p . An element of E/N_p will be denoted by x_p ($\equiv x + N_p$, $x \in E$) and the natural morphism from E onto E/N_p by π_p , $p \in \Gamma_E$. If r_E is the *spectral radius* of E and E is advertibly complete, one has [11, Theorem III, 6.1]

$$(1.1) \quad r_E(x) = \sup_p r_{E_p}(x_p), \quad x \in E.$$

2. RESULTS

For the sake of completeness we first state the following lemmas.

Lemma 2.1 [1, p. 126, Proposition 1(ii)]. *Let E be an algebra and $x \in E$ with $r_E(yx) = 0$, for all $y \in E$. Then $x \in J_E$. $\square$*

In [1] Lemma 2.1 is stated for Banach algebras, but its proof is, in fact, purely algebraic (remark due to A. Mallios).

Lemma 2.2 [14, Lemma 2]. *Let E be a Banach algebra and $f(\lambda)$, $\lambda \in \mathbb{C}$, a polynomial with coefficients in E . Then, for $k > 0$ one has*

$$(2.1) \quad r_E(f(1))^2 \leq \sup_{|\lambda|=k} r_E(f(\lambda)) \sup_{|\lambda|=k^{-1}} r_E(f(\lambda)). \quad \square$$

The following results extend analogous ones of [14] for Banach algebras. I am indebted to Professor A. Mallios whose remarks led to the statement of the next result and to Definition 2.5 (but see also [2, p. 316]).

Theorem 2.3. *Let E , F be lmc algebras with E being moreover *Q* and F advertibly complete. Suppose that Φ is a morphism of E in F . Then*

- (i) *the image (under $\text{id}_E \times \Phi$) of a null net in E converges, within the graph of Φ , to at most a topologically nilpotent element of F .*
- (ii) *If Φ is moreover surjective the preceding limit point belongs, in particular, to the radical J_F of F .*

Proof. (i) Let $\Gamma_F = \{q\}$ and (x_δ) be a net in E with $x_\delta \rightarrow 0$ and $\Phi(x_\delta) \rightarrow y = \Phi(x)$, $x \in E$. By the continuity of π_q we get

$$\Phi_q(x_\delta) \equiv (\pi_q \circ \Phi)(x_\delta) \rightarrow \pi_q(y) = y_q.$$

Now, for $\lambda \in \mathbb{C}$, take (cf. also [14, p. 489])

$$f_\delta(\lambda) = \Phi_q(\lambda x_\delta - x_\delta + x).$$

Then

$$r_{F_q}(f_\delta(\lambda)) \leq \|f_\delta(\lambda)\|_q \leq |\lambda| \|\Phi_q(x_\delta)\|_q + \|\Phi_q(x_\delta) - y_q\|_q.$$

Since E is Q , there is $p \in \Gamma_E$ with $r_E \leq p$ [15, (VI)]. Moreover, since Φ_q is a morphism, we conclude that

$$\begin{aligned} r_{F_q}(f_\delta(\lambda)) &= r_{F_q}(\Phi_q(\lambda x_\delta - x_\delta + x)) \leq r_E(\lambda x_\delta - x_\delta + x) \\ &\leq |\lambda| p(x_\delta) + p(x_\delta - x). \end{aligned}$$

On the other hand, $f_\delta(1) = y_q$, for all δ ; therefore (cf. (2.1))

$$r_{F_q}(y_q)^2 \leq (kp(x_\delta) + p(x - x_\delta))(k^{-1} \|\Phi_q(x_\delta)\|_q + \|\Phi_q(x_\delta) - y_q\|_q).$$

Now taking limits, first with respect to δ , and second for $k \rightarrow \infty$, we obtain $r_{F_q}(y_q) = 0$, for all $q \in \Gamma_F$. Hence (cf. (1.1)) $r_F(y) = 0$.

(ii) Now suppose that Φ is surjective and take $y' \in F$. Then, $y' = \Phi(x')$ for some $x' \in E$. Moreover, $x'x_\delta \rightarrow 0$ and

$$\Phi(x'x_\delta) = \Phi(x')\Phi(x_\delta) \rightarrow \Phi(x')\Phi(x) = y'y.$$

Thus, by repeating the argument applied in (i) we get $r_F(y'y) = 0$, for all $y' \in F$. Lemma 2.1 implies now $y \in J_F$. $\square$

An immediate consequence of the preceding theorem is the next.

Corollary 2.4. *Let E, F be lmc algebras such that E is moreover Q and F semisimple advertibly complete. Then every surjective morphism $\Phi: E \rightarrow F$ has a closed graph. $\square$*

Using any of the closed graph theorems for locally convex spaces (cf. [4, 6, 9]) we can get various conditions on the algebras E, F of Corollary 2.4 such that Φ is continuous. In this respect we have the following

Definition 2.5. A pair (E, F) of locally convex spaces will be called a *closed graph pair* whenever any linear map $\Phi: E \rightarrow F$, with closed graph, is continuous.

In view of Corollary 2.4, we now get

Theorem 2.6. *Let E be a Q -algebra and F a semisimple advertibly complete lmc algebra such that (E, F) is a closed graph pair. Then each surjective morphism $\Phi: E \rightarrow F$ is continuous. $\square$*

The above results are still valid for E a locally convex Q -algebra. An extension of Johnson's theorem is stated as follows.

Theorem 2.7. *Let (E, τ) be a semisimple Q -algebra. Let τ' be a second topology on E such that (E, τ') is a (semisimple) Q -algebra and $((E, \tau), (E, \tau')), ((E, \tau'), (E, \tau))$ are closed graph pairs. Then τ is uniquely determined.*

Proof. Every Q -algebra is advertibly complete (cf. §1). Thus the assertion follows by applying Theorem 2.6 for the identity map $\text{id}_E: (E, \tau) \rightarrow (E, \tau')$. $\square$

Corollary 2.8. *The following classes of semisimple lmc algebras have a uniquely determined topology:*

- (i) *Barrelled Pták Q -algebras.*
- (ii) *LFQ-algebras.*
- (iii) *[0, Remark 3] FQ-algebras.*

Proof. Clearly (iii) follows by (i). Let (E, τ) be a semisimple lmc algebra as in (i), (ii) resp. and τ' a second topology on E such that (E, τ') is a barrelled Pták Q -algebra, resp. an LFQ-algebra. Then $((E, \tau), (E, \tau')), ((E, \tau'), (E, \tau))$ are closed graph pairs according to [6, p. 301, Theorem 4], resp. [4, Corollary 1.2.20(ii)]. Thus applying Theorem 2.7 we get the conclusion. $\square$

3. APPLICATIONS

For the terminology applied in Proposition 3.1, see [11, Chapter V, §4].

Proposition 3.1. *Let X be a complex and 2nd countable manifold and K a compact Stein set in X . Let $O(K)$ denote the algebra (of germs) of holomorphic functions on K [11, Chapter V, 4(3)]. Then the usual topology on $O(K)$ is the only one making $O(K)$ an LFQ-algebra.*

Proof. According to [11] one has by definition that

$$O(K) = \varinjlim O(U_n),$$

where (U_n) , $n \in \mathbb{N}$, is a denumerable open basis of neighborhoods of K in X . Thus, $O(K)$ is equipped with the final locally convex topology induced on it by the natural maps $O(U_n) \rightarrow O(K)$, $n \in \mathbb{N}$, where $O(U_n)$ carries the compact-open topology from $C(U_n)$ (complex continuous functions on U_n). On the other hand, considering the algebra $\mathfrak{A}(\overline{U}_n)$ of all complex continuous functions on the closure $\overline{U}_n$ of U_n that are holomorphic on U_n , one further obtains

$$\varinjlim O(U_n) = \varinjlim \mathfrak{A}(\overline{U}_n),$$

up to a locally convex space isomorphism [11, p. 136]. Thus, one finally gets that $O(K)$ becomes an LFQ-algebra [11, p. 137]. Moreover, if $\mathfrak{M}(O(K))$ is the Gel'fand space of $O(K)$, then one has [11, Chapter V, (4.6)] $\mathfrak{M}(O(K)) = K$, with respect to a homeomorphism. Hence, $O(K)$ is semisimple in the sense that $\bigcap \{\ker(f) : f \in \mathfrak{M}(O(K))\} = \{0\}$ [11, Definition VIII, 3.2]. Now, because of commutativity, $O(K)$ is also Jacobson semisimple, so that the assertion follows by Corollary 2.8(ii). $\square$

Remark 3.2. The conclusion of Proposition 3.1 could also be derived by Corollary 2.8(i) since $O(K)$ is, in particular, an example of a Pták Q -algebra [12, Theorem 5.1] which is besides, barrelled, as an LF-space [8, p. 368].

Proposition 3.3. *Let $\mathcal{D}(X)$ be the algebra of test functions on a (locally compact) σ -compact finite-dimensional C^∞ -manifold X [11, Chapter IV, 4(2)]. Then the usual topology on $\mathcal{D}(X)$ is the only one making $\mathcal{D}(X)$ an LFQ-algebra.*

Proof. Let $C^\infty(X)$ be the F -algebra of all complex C^∞ -functions on X [11, Chapter IV, (4.19)]. Then

$$\mathcal{D}(X) = \{f \in C^\infty(X) : f \text{ has compact support}\}$$

is a subalgebra of $C^\infty(X)$. Let τ_∞ be the relative lmc topology on $\mathcal{D}(X)$ induced on it by $C^\infty(X)$. Now, by hypothesis, $X = \bigcup_n K_n$, with $K_n \subset X$ compact. For every $n \in \mathbb{N}$ consider the F -algebra

$$\mathcal{D}_{K_n}(X) = \{f \in \mathcal{D}(X) : \text{supp}(f) \subset K_n\}.$$

Then

$$\mathcal{D}(X) = \varinjlim \mathcal{D}_{K_n}(X)$$

up to a bijection [11, Chapter IV, (4.23)]. Equip $\mathcal{D}(X)$ with the final locally convex topology τ that makes the natural injections $\mathcal{D}_{K_n}(X) \rightarrow \mathcal{D}(X)$, $n \in \mathbb{N}$, continuous. Then, one obtains that $(\mathcal{D}(X), \tau)$ is an LF-algebra [11, Chapter IV, (4.25)]. We shall further prove that $(\mathcal{D}(X), \tau)$ is a semisimple Q -algebra. Let $\mathfrak{M}_\infty$ be the Gel'fand space of $(\mathcal{D}(X), \tau_\infty)$ and $\mathfrak{M}_\tau$ that of $(\mathcal{D}(X), \tau)$. Since $\tau_\infty \prec \tau$ we get that $\mathfrak{M}_\infty \subseteq \mathfrak{M}_\tau$ with respect to a continuous injection.

On the other hand, $\overline{(\mathcal{D}(X), \tau)} = C^\infty(X)$ [11, p. 133]; therefore,

$$(3.1) \quad \mathfrak{M}_\infty = X$$

[11, p. 145 (1.7), p. 127, Scholium 2.1]. Moreover, $(\mathcal{D}(X), \tau_\infty)$ is advertibly complete (cf. [11, Scholium IV, 4.1] as well as [16, p. 8]), hence by [11, Corollary III, 6.5] and the fact that each $f \in \mathcal{D}(X)$ has compact support we conclude that

$$r_{\mathcal{D}(X)}(f) = \sup_{\Phi \in \mathfrak{M}_\tau} |f(\Phi)| = \sup_{\Phi \in \mathfrak{M}_\infty} |f(\Phi)| = \sup\{|f(x)| : x \in X\} < \infty.$$

Hence, [11, Lemma III, 6.1] implies that $(\mathcal{D}(X), \tau)$ is a Q -algebra. Furthermore, because of (3.1), the Gel'fand map $\mathfrak{G}_\infty$ of $(\mathcal{D}(X), \tau)$ is 1-1. On the other hand, the relation between $\mathfrak{M}_\infty$, $\mathfrak{M}_\tau$ induces a map α between the lmc algebras $C_c(\mathfrak{M}_\tau)$, $C_c(\mathfrak{M}_\infty)$ (complex continuous functions on $\mathfrak{M}_\tau$, resp. $\mathfrak{M}_\infty$, with the compact-open topology). Thus, one obtains that the diagram

$$\begin{array}{ccc} C_c(\mathfrak{M}_\tau) & \xrightarrow{\alpha} & C_c(\mathfrak{M}_\infty) \\ \mathfrak{G}_\tau \uparrow & & \uparrow \mathfrak{G}_\infty \\ (\mathcal{D}(X), \tau) & \xrightarrow{\text{id}} & (\mathcal{D}(X), \tau_\infty) \end{array}$$

is commutative. As a result, one now gets that the Gel'fand map $\mathfrak{G}_\tau$ of $(\mathcal{D}(X), \tau)$ is 1-1, therefore $(\mathcal{D}(X), \tau)$ is (Jacobson) semisimple. Hence, the conclusion follows from Corollary 2.8(ii). $\square$

Let $E = \varinjlim E_\alpha$ be an inductive limit algebra [11, Definition IV, 1.1]. An involution $*$ on E is called *compatible* (with the respective inductive system) whenever each algebra E_α remains invariant under $*$. The involution on $\mathcal{D}(X)$ (cf. Proposition 3.3) defined by the complex conjugate is compatible.

In this respect, one now has

Theorem 3.4. *A compatible involution on a semisimple LFQ -algebra is automatically continuous.*

Proof. Let E be an LFQ -algebra with topology τ and involution $*$. Then E will be the inductive limit of an increasing sequence (E_n, τ_n) , $n \in \mathbb{N}$, of F -algebras such that $\tau|_{E_n} = \tau_n = \tau_{n+1}|_{E_n}$, $n \in \mathbb{N}$. If (p_m) , $m \in \mathbb{N}$, is the sequence of the algebra seminorms defining the topology τ_n , set

$$q_m(x) := p_m(x^*), \quad x \in E_n, m \in \mathbb{N}.$$

Then the sequence (q_m) , $m \in \mathbb{N}$, defines a new topology τ_n^* on E_n such that (E_n, τ_n^*) is an F -algebra and $\tau_{n+1}^*|_{E_n} = \tau_n^*$, $n \in \mathbb{N}$. Consider the inductive limit topology τ^* on E , i.e., the final lmc topology on E making the injections $(E_n, \tau_n^*) \rightarrow (E, \tau^*)$, $n \in \mathbb{N}$, continuous. Then, $\tau^*|_{E_n} = \tau_n^*$, so that (E, τ^*) is an LF -algebra and the map $*$: $(E, \tau) \rightarrow (E, \tau^*)$ a homeomorphism. Thus, (E, τ^*) is a Q -algebra since this is the case for (E, τ) . Corollary 2.8(ii) now yields the equivalence of τ , τ^* and hence the assertion. $\square$

It is easily seen that *any involution on a semisimple barrelled Pták Q -algebra is automatically continuous.*

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ADDED IN PROOF

Concerning §3 we note that another application of Theorem 2.7 is the algebra $\mathcal{K}(X)$, X a locally compact σ -compact space, of all $\mathbb{C}$ -valued continuous functions on X with compact support, endowed with the inductive limit topology τ [11, p. 127, 4(1)]. In fact, $(\mathcal{K}(X), \tau)$ is an LFQ -algebra (ibid.), the Gel'fand space of which contains all point evaluations δ_x , $x \in X$. Hence $\mathcal{K}(X)$ is semisimple and τ is the only topology making $\mathcal{K}(X)$ an LFQ -algebra.

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