RESTRICTIONS OF DISCRETE SERIES TO $\mathfrak{sl}(2, \mathbb{R})$

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ABSTRACT. We prove that the restriction to $\mathfrak{sl}(2, \mathbb{R})$ of the underlying Harish-Chandra module of a Discrete Series representation of a noncompact simple Lie group has no $\mathfrak{sl}(2)$ -irreducible subrepresentations unless it is holomorphic. We also give a characterization of the holomorphic Discrete Series.

Let G be a connected real simple Lie group. Fix a maximal compact subgroup K of G . Let $\mathfrak{g}(\mathbb{C})$ denote the complex Lie algebra of G (respectively, K). Hence we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Thus, $\mathfrak{p}$ is K -invariant under the restriction of the adjoint representation of G to K and $\mathfrak{p}$. Let $T \subseteq K$ be a maximal torus. Henceforth we assume that T is a Cartan subgroup of G . Let $\mathfrak{t}$ be the complex Lie algebra of T . Under these hypotheses Harish-Chandra in [HC2] has given a classification of the square-integrable representations of G in terms of the regular characters of T . Let $\Phi(\mathfrak{g}, \mathfrak{t})$ denote the root system of the pair $(\mathfrak{g}, \mathfrak{t})$. A root of the pair $(\mathfrak{g}, \mathfrak{t})$ is called compact (noncompact) if its root space is contained in $\mathfrak{k}$ (respectively, $\mathfrak{p}$). We now fix a noncompact root γ . Then the root vectors of γ span a copy $\mathfrak{h}_\gamma$ of $\mathfrak{sl}(2, \mathbb{C})$. Moreover $\mathfrak{h}_\gamma$ is invariant under the conjugation of $\mathfrak{g}$ with respect to the Lie algebra of G . Let $\mathfrak{s}_\gamma$ be the real form of $\mathfrak{h}_\gamma$ associated to the conjugation of $\mathfrak{g}$ with respect to the Lie algebra of G restricted to $\mathfrak{h}_\gamma$. Let H_γ be the connected subgroup of G corresponding to $\mathfrak{s}_\gamma$. Let (π, V) be the Harish-Chandra module underlying a square-integrable, irreducible representation of G . We then have

Proposition 1. *If the restriction of (π, V) to $\mathfrak{h}_\gamma$ contains an irreducible $\mathfrak{h}_\gamma$ -submodule, then G/K is a Hermitian symmetric space and (π, V) corresponds to a holomorphic square-integrable representation.*

A corollary to Proposition 1 is

Corollary. *If G is as above, G/K is not Hermitian, and H is a semisimple subgroup of G such that its symmetric space is Hermitian, then (π, V) restricted to $\mathfrak{h}$ has no $\mathfrak{h}$ -holomorphic irreducible subrepresentations.*

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For the next proposition let (τ, W) be an irreducible square-integrable representation of G . Let V be the subspace of all T -finite vectors in W . We have

Proposition 2. *Assume that V is T -admissible. Then G/K is a Hermitian symmetric space and (τ, W) is a holomorphic square-integrable representation of G .*

Remark. That is, the multiplicity of each T -type in V is finite.

Let us recall that in [HC1] it is proved that a holomorphic discrete series representation is T -finite.

Let us briefly recall a few definitions.

(a) A system of positive roots Ψ for the pair $(\mathfrak{g}, \mathfrak{t})$ is called *holomorphic* if the sum of two noncompact roots in Ψ never is a root and if the sum of any compact root with a noncompact root in Ψ , if it is a root, is in Ψ . In [HC1] it is proved that there exists a holomorphic system of positive roots if and only if G/K has a G -invariant complex structure.

(b) The $(\mathfrak{g}, K)$ -module underlying a square-integrable representation is called *holomorphic* if it is a lowest weight module for a holomorphic system of positive roots in $\Phi(\mathfrak{g}, \mathfrak{t})$.

Let us recall that Martens in [M] has proven that if (π, V) is a holomorphic square-integrable representation of G and H is a semisimple subgroup of G containing a compact Cartan subgroup of G , then π restricted to H is a discrete Hilbert sum of irreducible square-integrable holomorphic representations and each factor occurs with finite multiplicity. She also provides a complete description of the multiplicities in terms of the Harish-Chandra classification of square-integrable representations.

The proof of Proposition 1 will be done in a sequence of lemmas.

For any Lie algebra $\mathfrak{s}$, let $U(\mathfrak{s})$ denote its universal enveloping algebra, and for any vector space E , let $S(E)$ denote its symmetric algebra.

Let V be any $(\mathfrak{g}, K)$ -module and let W be a finite-dimensional K -invariant subspace of V . Let I be the annihilator of W in $U(\mathfrak{g})$. Therefore, I is a left ideal and I is K -invariant under the adjoint action of K in $U(\mathfrak{g})$. We now consider J , the homogeneous ideal of the graded algebra $S(\mathfrak{g})$ associated to the usual filtration on $U(\mathfrak{g})$. We have that

- (1) J is K -invariant under the action of Ad_G restricted to K and lifted to $S(\mathfrak{g})$.
- (2) J is a homogeneous ideal.
- (3) $J \cap S(\mathfrak{k})$ is a finite codimension ideal in $S(\mathfrak{k})$.

In fact, (1) and (2) are clear. (3) is true because W is finite-dimensional, hence W is annihilated by a finite codimension ideal in $U(\mathfrak{k})$, and the filtration on $U(\mathfrak{k})$ is compatible with the filtration on $U(\mathfrak{g})$.

Since $\mathfrak{g}$ is a semisimple complex Lie algebra, the Killing form of $\mathfrak{g}$ is an $\text{Ad}(G)$ -invariant, nondegenerate, symmetric bilinear form, which allows us to identify $\mathfrak{g}$ with its dual vector space in a G -equivariant manner. From now on, we will use this identification. Hence, we may think of $S(\mathfrak{g})$ as the ring of polynomial functions on $\mathfrak{g}$. Let $V(J)$ denote the zero set of the ideal J . We have that

- (4) $V(J)$ is contained in $\mathfrak{p}$.

- (5) $V(J)$ is $\text{Ad}_G(K)$ -invariant.
 (6) If $V(J) = 0$ and V is irreducible, then V is a finite-dimensional vector space.

In fact, since $J \cap S(\mathfrak{k})$ has finite codimension in $S(\mathfrak{k})$ and the radical of any homogeneous finite codimension ideal is the maximal ideal corresponding to the origin, the radical of J contains $\mathfrak{k}S(\mathfrak{k})$; therefore $V(J)$ is contained in $\mathfrak{p}$ and we have verified (4). (5) is trivial. If $V(J) = 0$, then J (being homogeneous) has finite codimension in $S(\mathfrak{g})$. Therefore, I has finite codimension in $U(\mathfrak{g})$. Thus, $U(\mathfrak{g})W$ is a finite-dimensional $\mathfrak{g}$ -invariant subspace of V . Since V is irreducible we have (6).

- (7) If V has an infinitesimal character, then $V(J)$ is contained in the nilpotent cone.

In fact, V is annihilated by a maximal ideal of the center of $U(\mathfrak{g})$, so a theorem of Kostant [K] proves that $V(J)$ is contained in the nilpotent cone.

- (8) We now fix a system of positive roots Ψ in the root system of the pair $(\mathfrak{g}, \mathfrak{t})$. Let β be the highest noncompact root in Ψ (since $\mathfrak{g}$ is simple β is unique). For each root α , X_α will denote a nonzero root vector in the root space corresponding to α .

Lemma 9. *Let C be a nontrivial, $\text{Ad}_G(K)$ -invariant, and nilpotent cone in $\mathfrak{p}$. Then either X_β or $X_{-\beta}$ is in C . Moreover, if G/K is not Hermitian symmetric, then both root vectors are in C .*

For a proof of the first (second) statement cf. [Wo] or [Vo] (respectively, [V1]), where there is a proof of: If G/K is not Hermitian symmetric, then the compact Weyl group acts transitively on the set of noncompact roots of a given length. If G/K is Hermitian symmetric, then the compact Weyl group acts transitively on the set of noncompact roots of a given length in a holomorphic system of positive roots.

Corollary 10. (i) *If G/K is not a Hermitian symmetric space, then all the root vectors corresponding to the long noncompact roots are in C .*

(ii) *Assume that G/K is a Hermitian symmetric space and choose Ψ so that it is holomorphic. Then if X_β ($X_{-\beta}$) is in C we have that all the root vectors corresponding to the noncompact long roots in Ψ (respectively $-\Psi$) are in C .*

We now fix a nilpotent element X in $\mathfrak{g}$. Let

$$V_X = \{v \in V : X^m v = 0 \text{ for some } m \gg 0\}.$$

Thus, V_X is nonzero if and only if there exists some nonzero vector annihilated by X .

Lemma 11. V_X is a $\mathfrak{g}$ -invariant subspace.

Proof. In $U(\mathfrak{g})$ we have the identity

$$(12) \quad X^k b = b X^k + \sum_{i=1}^k \binom{k}{i} \text{ad}(X)^{(i)}(b) X^{k-i}.$$

Therefore the lemma follows. $\square$

Lemma 13. *Let V be an irreducible infinite-dimensional $(\mathfrak{g}, K)$ -module. If G/K is not a Hermitian symmetric space, then $V_{X_\sigma} = 0$ for every noncompact root σ .*

Proof. The fact that G/K is not a Hermitian symmetric space implies that any two noncompact roots of the same length are conjugate under an element of the compact Weyl group [V1]. Since V is a $(\mathfrak{g}, K)$ -module we have that $V_{X_\sigma} = \pi(k)V_{X_\gamma}$ if $\text{Ad}(k)X_\gamma = X_\sigma$. Thus, “ V_{X_σ} is nonzero” is equivalent to “ V_{X_γ} is nonzero” for every noncompact root of the same length. Now, if V_{X_β} were nonzero (β as in (8)) the irreducibility of V would imply that V_{X_β} is equal to V . Let W be the subspace $U(\mathfrak{k})v$, where v is a fixed nonzero vector in V . Since V is a $(\mathfrak{g}, K)$ -module, W is a finite-dimensional K -invariant subspace of V and X_β^n belongs to the annihilator I of W in $U(\mathfrak{g})$ for some $n \gg 0$. Since the Killing form evaluated on the pair $(X_\beta, X_{-\beta})$ is nonzero we have that $X_{-\beta}$ does not belong to the variety $V(J)$ of the homogeneous ideal J associated to I . However, since V is infinite-dimensional, (7) says that $V(J)$ is nontrivial and Lemma 9 implies that both vectors $X_{\pm(\beta)}$ are in $V(J)$, contradiction! If V_{X_γ} were nonzero for some noncompact short root γ , the fact that G/K is not Hermitian symmetric implies that there exists a long noncompact root τ such that $\tau + \gamma$ is a root (cf. comment after Lemma 9). Hence, the Killing form evaluated at (X_τ, X_γ) is nonzero. Thus, X_β does not belong to $V(J)$ for the finite-dimensional K -invariant subspace spanned for a nonzero vector annihilated by X_γ . On the other hand our hypothesis makes true the conclusion of Lemma 9. Contradiction! $\square$

Corollary. *If G/K is not a Hermitian symmetric space and (π, V) is the underlying Harish-Chandra module of a square-integrable representation of G , then (π, V) restricted to $\mathfrak{s}_\gamma$ does not contain any irreducible $\mathfrak{s}_\gamma$ -submodule for any noncompact root γ .*

Proof. In fact, in [V2] we have proven that any irreducible H_γ -subrepresentation of the unitary model of V is integrable. For $\text{SL}(2, \mathbb{R})$ (locally isomorphic to H_γ), in any irreducible square-integrable representation either X_γ or $X_{-\gamma}$ acts as a locally nilpotent operator. Hence, if V had a $\mathfrak{s}_\gamma$ -invariant, irreducible subspace taking its closure in a unitary model of V we conclude that either X_γ or $X_{-\gamma}$ acts as a locally nilpotent operator, a contradiction. $\square$

Lemma 14. *If G/K is a Hermitian symmetric space, V an infinite-dimensional irreducible $(\mathfrak{g}, K)$ -module, then at most one of the two subspaces $V_{\pm\beta}$ is nonzero.*

Proof. If both subspaces were nonzero, then both would be equal to V . Hence, both vectors $X_{\pm\beta}$ would act as locally nilpotent operators. Fix any nonzero vector v in V and consider W , the $U(\mathfrak{k})$ -subspace spanned by v . Then $X_{\pm\beta}$ would belong to J (J being the homogeneous ideal associated to W). Hence $X_{\pm\beta}$ would not belong to $V(J)$ (the Killing form evaluated at $(X_\beta, X_{-\beta})$ is nonzero), which contradicts (9). $\square$

Lemma 15. *Assume that G/K is Hermitian, and that V is an irreducible infinite-dimensional $(\mathfrak{g}, K)$ -module. Then if V_{X_σ} is nonzero for some noncompact root σ , we have that V is a highest weight module with respect to a holomorphic system of positive roots.*

Proof. We first consider when V_{X_γ} is nonzero for some noncompact long root. We may assume that σ is positive with respect to a holomorphic chamber Ψ .

Since $X_\gamma \in \text{Ad}(K)X_\sigma$ for every noncompact long root γ in Ψ , we have that V_{X_γ} is equal to V for every noncompact long root γ in Ψ . Since V is a $(\mathfrak{g}, K)$ -module we also have $V_{X_\alpha} = V$ for every compact root α . If all the noncompact roots have the same length we are done because a nilpotent Lie algebra acting locally nilpotently on a vector space V has a nontrivial eigenvector (Engel's theorem). Thus, V is a Ψ -highest weight module, equivalently, a $-\Psi$ -lowest weight module. Otherwise, we now show that V_{X_γ} is nonzero for every noncompact short root in Ψ . It is enough to prove the statement for some noncompact short root in Ψ . Let $\mathfrak{p}_+$ be the subspace of $\mathfrak{p}$ spanned by the root subspaces corresponding to the noncompact roots in Ψ . Since Ψ is a holomorphic system we have that $\mathfrak{p}_+$ is an abelian, $\text{Ad}(K)$ -invariant, Lie subalgebra of $\mathfrak{g}$. Also, the fact that $\mathfrak{g}$ is simple implies that the K -action in $\mathfrak{p}_+$ is irreducible. Hence, there exists a noncompact long root τ in Ψ and a compact root α so that $\gamma = \tau + \alpha$ is a short root. In fact, γ is a noncompact short root in Ψ . Let us prove that V_{X_γ} is nonzero. Pick v a nonzero vector in V so that $X_\alpha v = 0$. Then $X_\tau^k v = 0$ and $X_\tau^{k-1} v \neq 0$ for some $k > 0$. The identity (12) implies that

$$\begin{aligned} X_\tau^k X_\alpha(v) &= X_\alpha X_\tau^k(v) + k[X_\tau, X_\alpha]X_\tau^{k-1}(v) \\ &+ \sum_{2 \leq i \leq k} c_i \text{ad}(X_\tau)^{(i-2)}([X_\tau, [X_\tau, X_\alpha]])X_\tau^{k-i}(v). \end{aligned}$$

The fact that G/K is Hermitian symmetric implies that $[X_\tau, X_\gamma] = 0$ and since $X_\gamma = [X_\tau, X_\alpha]$, we have that $X_\gamma(X_\tau^{k-1}(v)) = 0$. We now consider the case that V_{X_σ} is nonzero for some short noncompact root σ . As before, we fix a holomorphic system Ψ so that σ belongs to Ψ . We need to show that V_{X_γ} is nonzero for some noncompact long root γ in Ψ . Since $\mathfrak{p}_+$ is an irreducible K -module, there exists a compact root α so that $\gamma = \sigma + \alpha$ is a long root; we now proceed as before.

Corollary. *Let G/K be Hermitian symmetric, (π, V) the underlying $(\mathfrak{g}, K)$ -module of a nonholomorphic square-integrable representation. Then (π, V) has no $\mathfrak{h}_\gamma$ -irreducible subrepresentations for any noncompact root γ .*

Therefore we have completed the proof of Proposition 1. $\square$

Proof of Proposition 2. Let (π, Z) be the Harish-Chandra module underlying W . Hence $Z \subseteq V$. For a character λ of T let $V[\lambda]$ be the subspace of all vectors in V of type λ . That is, $V[\lambda] := \{v \in V : \tau(t)v = \lambda(t)v \text{ for every } t \text{ in } T\}$. By hypothesis, $V[\lambda]$ is finite-dimensional. Thus, $Z[\lambda] := V[\lambda] \cap Z$ is a finite-dimensional vector space. We fix λ so that $Z[\lambda]$ is nonzero. We also fix a noncompact root γ and, as before, let $\mathfrak{h}_\gamma$ be the associated copy of $\mathfrak{sl}(2, \mathbb{C})$ in $\mathfrak{g}$. Let $\Omega := H_\gamma^2 + H_\gamma + 2X_{-\gamma}X_\gamma$ be the Casimir operator of $\mathfrak{h}_\gamma$. It is clear that $\pi(\Omega)$ maps $Z[\lambda]$ into itself. Since (τ, W) is a unitary representation, a $\pi(\Omega)$ is symmetric operator in Z . Therefore $\pi(\Omega)$ has a nontrivial eigenvector v in $Z[\lambda]$. Since Ω spans the center $\mathcal{Z}(U(\mathfrak{h}_\gamma))$ of $U(\mathfrak{h}_\gamma)$, we have that $\pi(U(\mathfrak{h}_\gamma))v$ is a $(\mathfrak{h}_\gamma, T \cap H_\gamma)$ -module that is finitely generated and $\mathcal{Z}(U(\mathfrak{h}_\gamma))$ -finite. By a theorem of Harish-Chandra $\pi(U(\mathfrak{h}_\gamma))v$ has an irreducible submodule (for a proof cf. [Vo1, Corollary 5.4.16]). Thus, (π, Z) has an irreducible $\mathfrak{h}_\gamma$ -submodule. Proposition 1 implies that (π, Z) is holomorphic. $\square$

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