MATRIX COMPLETIONS, NORMS, AND HADAMARD PRODUCTS

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ABSTRACT. Let $M_{m,n}$ (respectively, H_n) denote the space of $m \times n$ complex matrices (respectively, $n \times n$ Hermitian matrices). Let $S \subset H_n$ be a closed convex set. We obtain necessary and sufficient conditions for $X_0 \in S$ to attain the maximum in the following concave maximization problem:

$$\max\{\lambda_{\min}(A + X): X \in S\}$$

where $A \in H_n$ is a fixed matrix. Let $\circ$ denote the Hadamard (entrywise) product, i.e., given matrices $A = [a_{ij}]$, $B = [b_{ij}] \in M_{m,n}$ we define $A \circ B = [a_{ij}b_{ij}] \in M_{m,n}$.

Using the necessary and sufficient conditions mentioned above we give elementary and unified proofs of the following results.

(a) For any $A \in M_{n,n}$

$$\omega(A) \equiv \max\{|x^*Ax|: x \in C^n, x^*x = 1\} \leq 1$$

if and only if there is a matrix $Z \in H_n$ such that

$$\begin{pmatrix} I + Z & A \\ A^* & I - Z \end{pmatrix} \geq 0.$$

(b) For any $A \in M_{m,n}$

$$\max\{\|A \circ B\|_\infty: \|B\|_\infty \leq 1\} \leq 1$$

if and only if there are matrices $P \in H_m$, $Q \in H_n$ such that

$$\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \geq 0, \quad P \circ I \leq I, \quad Q \circ I \leq I.$$

(c) For any $A \in M_{n,n}$

$$\max\{\omega(A \circ B): \omega(B) \leq 1\} \leq 1$$

if and only if there is a matrix $P \in H_n$ such that

$$\begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \geq 0, \quad P \circ I \leq I.$$

We also consider other norms that can be represented in this way and show that if a norm can be represented in this way then so can its dual.

1. INTRODUCTION AND PRELIMINARIES

Let $M_{m,n}$ denote the space on $m \times n$ complex matrices and let $M_n = M_{n,n}$. Let H_n denote the space of complex Hermitian matrices. We use $A \geq$

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B to mean that $A - B$ is positive semidefinite. Given $A \in M_{m,n}$ define $|A| \equiv (A^*A)^{1/2}$. We use $A^\dagger$ to denote the Moore-Penrose inverse of A . Given $A, B \in M_{m,n}$, the *Hadamard product* (or Schur product) of A and B is $A \circ B = [a_{ij}b_{ij}] \in M_{m,n}$. We use $e_n \in R^n$ to denote the vector of 1's (when the dimension is clear from the context, we use e). We will consider the following norms:

$$\begin{aligned}\|A\|_\infty &\equiv \max\{x^*Ay: x \in C^n, y \in C^m, x^*x \leq 1, y^*y \leq 1\}, \\ \|A\|_{H,\infty} &\equiv \max\{\|A \circ B\|_\infty: B \in M_{m,n}, \|B\| \leq 1\}, \\ \omega(A) &\equiv \max\{|x^*Ax|: x \in C^n, x^*x \leq 1\}, \\ \|A\|_{H,\omega} &\equiv \max\{\omega(A \circ B): B \in M_n, \omega(B) \leq 1\}.\end{aligned}$$

The first two norms are defined on $M_{m,n}$, the second two are defined on M_n .

It is a well-known and useful fact (that follows easily from the above definition) that $\|A\|_\infty \leq 1$ if and only if

$$\begin{pmatrix} I & A \\ A^* & I \end{pmatrix} \geq 0$$

and that $\|A\|_\infty = 1$ if and only if the given block matrix is positive semidefinite and singular. In this paper we will generalize this result to the norms $\omega(\cdot)$ in §2, and $\|\cdot\|_{H,\infty}$ and $\|\cdot\|_{H,\omega}$ in §3. The results in §§2 and 3 are not entirely new, however, they are more detailed than existing results and the proofs are constructive and are simpler than the original proofs. These extra details allow us to prove some new result as corollaries. In §4 we consider generalizations of these results and the corresponding results for dual norms.

Our approach to these characterization problems is to consider the optimality conditions for an appropriate concave maximization problem. These conditions are given in Lemma 1.4. Lemma 1.1 allows us to interpret these conditions in terms of Hadamard products.

There are other results in matrix theory that have been proved by considering the optimality conditions for an appropriate optimization problem. For example, it was shown in [10, p. 230 and Theorem 3.5] that any positive definite matrix A may be written as $A = C\Lambda C$ where C is a correlation matrix and Λ is diagonal. It was shown in [8, Theorem 2.6] that if $A \in M_{m,n}$ is entrywise nonnegative then

$$\|A\|_\infty = \min\{\|(BB^*) \circ I\|_\infty \|(C^*C) \circ I\|_\infty: B \circ C = A\}.$$

Lemma 1.1. *Let $A, B \in H_n$ be positive semidefinite. The following are equivalent:*

- (a) $AB = 0$.
- (b) $\text{tr } AB = 0$.
- (c) $e^*(A \circ B^T)e = 0$.

Proof. It is immediate that (a) implies (b). To see that (b) implies (a) note that both statements are invariant under a unitary similarity, so we may assume that A is diagonal. The result is now clear, because the main diagonal entries of B are nonnegative and the i, i entry is zero if and only if the i th row and column of B is also 0. The equivalence of (b) and (c) follows from the fact

$$\text{tr } AB = \sum_{i,j} a_{ij} \bar{b}_{ij} = \text{tr } e^*(A \circ B^T)e. \quad \square$$

The Schur Product Theorem states that if $A, B \in H_n$ is positive semidefinite then so is $A \circ B$. The next lemma is a strengthening of the Schur Product Theorem.

Lemma 1.2. *Let $A, B \in H_n$ be positive semidefinite. Then $A \circ B$ is singular if and only if there is a nonzero diagonal matrix D such that $ADB^T D^* = 0$.*

Proof. By the Schur Product Theorem we know that $A \circ B$ is positive semidefinite. So if $A \circ B$ is singular then there is a nonzero vector x such that $x^*(A \circ B)x = 0$. Let $D = \text{diag}(x)$. Then

$$\begin{aligned} 0 &= x^*(A \circ B)x = \text{tr}(A \circ B)xx^* \\ &= \text{tr}((xx^*)^T \circ A)B^T = \text{tr}(D^*AD)B^T = \text{tr}A(D^*BD) \end{aligned}$$

and hence by Lemma 1.1 we have $A(D^*BD) = 0$. For the third equality we have used the fact (which can be verified computationally) that for any matrices $X, Y, Z \in M_n$

$$\text{tr}(X \circ Y)Z = \text{tr}(X \circ Z^T)Y^T.$$

To prove the converse reverse this argument. $\square$

Given a convex cone $K \subset H_n$ define its polar cone K^* by

$$K^* = \{A \in H_n : \text{tr}AB^* \leq 0 \text{ for all } B \in K\}.$$

We state the following theorem of the alternative as we will use it in an essential way twice. It is an analog in H_n of a standard theorem of the alternative for convex sets in R^n and can be derived in the same way (see, e.g., [7, §3.2, Theorem 6]).

Theorem 1.3. *Let $K \subset H_n$ be a convex cone and $W \subset H_n$ be a compact set. Then exactly one of the following holds:*

- (a) *There is a matrix $A \in K$ such that $\text{tr}AB > 0$ for all $B \in W$.*
- (b) *There are $\lambda_i > 0$ and $B_i \in W$ such that $\sum_{i=1}^k \lambda_i B_i \in K^*$.*

Optimality conditions similar to those in the next lemma are given in [12, 3].

Lemma 1.4. *Let S be a closed convex subset of H_n , and let $A \in H_n$ and $X_0 \in S$ be given. Let K be the cone of feasible directions at X_0 . Then the following are equivalent:*

- (a) $\lambda_{\min}(A + X_0) \geq \lambda_{\min}(A + X)$ for all $X \in S$.
- (b) *There is a nonzero positive semidefinite matrix $M \in K^*$ such that*

$$(A + X_0)M = \lambda_{\min}(A + X_0)M.$$

If S is a subspace of H_n then $K^ = S^\perp$ independent of X_0 .*

Proof. Without loss of generality we may assume that $\lambda_{\min}(A + X_0) = 0$. Let

$$N = \{y \in C^n : (A + X_0)y = 0 \text{ and } y^*y = 1\}$$

and

$$C = \{yy^* : y \in N\}.$$

Then C is a compact set. Also,

$$\lambda_{\min}(A + X_0) \geq \lambda_{\min}(A + X) \quad \text{for all } X \in S$$

if and only if there is no $X \in S$ such that

$$\operatorname{tr}(X - X_0)yy^* = y^*(X - X_0)y > 0 \quad \text{for all } y \in N,$$

or equivalently

$$(1.1) \quad \nexists D \in K \text{ such that } \operatorname{tr} DY > 0 \text{ for all } Y \in C.$$

However, because K is a convex cone and C is a compact set, the statement (1.1) is equivalent, by Theorem 1.3, to

$$\exists y_i \in N, \lambda_i > 0 \text{ such that } 0 \neq \sum_{i=1}^k \lambda_k y_i y_i^* \in K^*.$$

This completes the proof as M is a nonzero positive semidefinite matrix such that $(A + X_0)M = 0$ if and only if $M = \sum_{i=1}^k \lambda_i y_i y_i^*$ for some $k > 0$, $y_i \in N$, and $\lambda_i > 0$. $\square$

2. THE NUMERICAL RADIUS

In this section we give an elementary proof of a useful characterization of matrices that have numerical radius at most 1. Part (a) of Theorem 2.1 is due to Ando [1].

Theorem 2.1. *Let $A \in M_n$.*

(a) *$\omega(A) \leq 1$ if and only if there is a matrix $Z \in H_n$ such that*

$$(2.1) \quad \begin{pmatrix} I + Z & A \\ A^* & I - Z \end{pmatrix} \geq 0.$$

(b) *Suppose that $Z \in H_n$ is such that*

$$\begin{pmatrix} I + Z & A \\ A^* & I - Z \end{pmatrix} \geq 0.$$

Then $\omega(A) = 1$ if and only if there is a vector $x \in C^n$ and a scalar $\alpha \in C$ with $x^*x = |\alpha| = 1$ such that

$$\begin{pmatrix} I + Z & A \\ A^* & I - Z \end{pmatrix} \begin{pmatrix} x \\ \alpha x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In this case $|x^*Ax| = 1$.

(c) *$\omega(A) = 1$ if and only if there is a $2n \times 2n$ matrix*

$$(2.2) \quad \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{11} \end{pmatrix} \geq 0, \quad Y_{ij} \in M_n,$$

and a matrix $Z \in H_n$ such that

$$B \equiv \begin{pmatrix} I + Z & A \\ A^* & I - Z \end{pmatrix} \geq 0$$

and

$$(2.3) \quad \begin{pmatrix} I + Z & A \\ A^* & I - Z \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{11} \end{pmatrix} = 0.$$

Let Y be any such matrix, and let $\text{rank } Y = k$. Then there is a matrix $R \in M_{n,k}$ with nonzero columns and a diagonal unitary matrix $D \in M_k$ such that

$$\begin{pmatrix} R \\ RD \end{pmatrix} \begin{pmatrix} R \\ RD \end{pmatrix}^* = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{11} \end{pmatrix}$$

and for any column r_i of R we have $r_i \neq 0$ and $|r_i^* A r_i| = r_i^* r_i$.

The main content of this theorem can be restated as

$$(2.4) \quad -\omega(A) = \max_{Z \in H_n} \lambda_{\min} \begin{pmatrix} Z & A \\ A^* & -Z \end{pmatrix}.$$

In [1] Ando proves more than part (a) of the theorem. He shows that if $\omega(A) \leq 1$ then the set

$$\left\{ Z \in H_n : \begin{pmatrix} I+Z & A \\ A^* & I-Z \end{pmatrix} \geq 0 \right\}$$

has a minimal and maximal element (with respect to the partial order $\geq$ on H_n). It appears that our approach does not yield this additional information.

Proof. We will first show that if there is a matrix Z for which (2.1) holds then $\omega(A) \leq 1$. Let Z be such a matrix and take any $x \in C^n$ and $\alpha \in C$ with $x^* x = |\alpha| = 1$. Then

$$(2.5) \quad 0 \leq \begin{pmatrix} x \\ \alpha x \end{pmatrix}^* \begin{pmatrix} I+Z & A \\ A^* & I-Z \end{pmatrix} \begin{pmatrix} x \\ \alpha x \end{pmatrix} = 2 + 2\text{Re}(\alpha x^* A x).$$

Hence $\omega(A) \leq 1$. This calculation also establishes (b).

Now we will prove (c) and the other direction of (a) by considering

$$(2.6) \quad c(A) \equiv \max_{Z \in H_n} \lambda_{\min} \left(\begin{pmatrix} I+Z & A \\ A^* & I-Z \end{pmatrix} \right).$$

Let us first consider the case $c(A) = 0$. The maximization is over the subspace $S = \{Z \oplus (-Z) : Z \in H_n\}$. It is easy to show that

$$S^\perp = \left\{ \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{11} \end{pmatrix} \in H_{2n} : Y_{ii} \in M_n \right\}.$$

So by Lemma 1.4 this maximum is 0 and is attained at the matrix Z if and only if there is a matrix of the form (2.2) such that (2.3) holds. Let Y be such a matrix and let $\text{rank } Y = k$. Then, because $Y \geq 0$,

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{11} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}^*$$

for some matrices $R_i \in M_{n,k}$. Because $R_1 R_1^* = R_2 R_2^*$, there is a unitary matrix $U \in M_k$ such that $R_1 U = R_2$. Let $U = V D V^*$ where V, D are unitary and D is diagonal. Let $R = R_1 V$. Then

$$0 = \begin{pmatrix} I+Z & A \\ A^* & I-Z \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix},$$

and hence

$$0 = \begin{pmatrix} I+Z & A \\ A^* & I-Z \end{pmatrix} \begin{pmatrix} R_1 V \\ R_2 V \end{pmatrix} = \begin{pmatrix} I+Z & A \\ A^* & I-Z \end{pmatrix} \begin{pmatrix} R \\ RD \end{pmatrix}.$$

Because $\text{rank } Y = k$, every column of R must be nonzero. By part (b) it follows that $|r_i^* A r_i| = r_i^* r_i$ and that $\omega(A) = 1$. We have now established (c) and shown that if $c(A) = 0$ then $\omega(A) = 1$.

Now let us prove the ‘only if’ part of (a). Suppose that there is no matrix $Z \in H_n$ for which (2.1) holds. Then $c(A) < 0$. Straightforward manipulations show that if $c(A) < 0$ then $c([1 - c(A)]^{-1}A) = 0$. So by the preceding analysis, $\omega([1 - c(A)]^{-1}A) = 1$ or, equivalently, $\omega(A) = 1 - c(A)$. (The identity (2.4) follows from this.) So since $c(A) < 0$, we have $\omega(A) = 1 - c(A) > 1$. This proves the ‘only if’ part of (a). $\square$

The power inequality for the numerical radius states that if $\omega(A) \leq 1$ then $\omega(A^k) \leq 1$ for any positive integer k . (It is not true, in general, that $\omega(AB) \leq \omega(A)\omega(B)$.) This inequality follows easily from Theorem 2.1. For if $\omega(A) \leq 1$ then there is a Hermitian contraction Z such that

$$\begin{pmatrix} I + Z & A \\ A^* & I - Z \end{pmatrix} \geq 0,$$

and hence there is a contraction C such that

$$A = (I + Z)^{1/2} C (I - Z)^{1/2}.$$

So for any positive integer k

$$\begin{aligned} A^k &= (I + Z)^{1/2} C [(I + Z)^{1/2} (I - Z)^{1/2} C]^{k-1} (I - Z)^{1/2} \\ &= (I + Z)^{1/2} C [(I - Z^2)^{1/2} C]^{k-1} (I - Z)^{1/2}. \end{aligned}$$

Because C and Z are contractions and Z is Hermitian, $C[(I - Z^2)^{1/2} C]^{k-1}$ is also a contraction, and hence

$$\begin{pmatrix} I + Z & A^k \\ (A^k)^* & I - Z \end{pmatrix} \geq 0,$$

from which it follows that $\omega(A^k) \leq 1$.

3. HADAMARD PRODUCTS

In this section we consider the norms $\|\cdot\|_{H,\infty}$ and $\|\cdot\|_{H,\omega}$. The results in Theorem 3.1 can be derived from this in Theorem 3.2 by using the following identity [2, Lemma 8]:

$$\|A\|_{H,\infty} = \left\| \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \right\|_{H,\omega}$$

(the 0 blocks are $n \times n$). However, it is simpler to prove Theorem 3.1 directly. Theorem 3.1(a) is due to Haagerup (unpublished); it was proved, in a more general form (involving partial matrices), in [13, Theorem 3.2]; it was also derived by Ando and Okubo, using the identity above, from the main result in [2], which is Theorem 3.2(a). All these proofs are based on Arvensen’s Extension Theorem for positive maps on C^* algebras.

Theorem 3.1. *Let $A \in M_{m,n}$.*

(a) *$\|A\|_{H,\infty} \leq 1$ if and only if there are matrices $P \in H_m$ and $Q \in H_n$ such that*

$$(3.1) \quad \begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \geq 0, \quad P \circ I \leq I, \quad Q \circ I \leq I.$$

(b) Suppose that $P \in H_m$, $Q \in H_n$ are such that (3.1) holds. Then $\|A\|_{H,\infty} = 1$ if and only if there are matrices $D_1 \in M_m$, $D_2 \in M_n$, and $Y \in M_{m,n}$ with D_1 and D_2 diagonal and such that

$$(3.2) \quad \begin{pmatrix} D_1 & Y \\ Y^* & D_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \begin{pmatrix} D_1 & Y \\ Y^* & D_2 \end{pmatrix} = 0,$$

$$(3.3) \quad D_1(P \circ I - I) = 0, \quad D_2(Q \circ I - I) = 0.$$

For any such D_1 , D_2 , and Y

$$\sigma_1(D_1^{1/2\uparrow} \bar{Y} D_2^{1/2\uparrow}) = 1 \quad \text{and} \quad \sigma_1(A \circ [D_1^{1/2\uparrow} \bar{Y} D_2^{1/2\uparrow}]) = 1.$$

In particular,

$$y^*(A \circ [D_1^{1/2\uparrow} \bar{Y} D_2^{1/2\uparrow}])z = \sqrt{y^*y} \sqrt{z^*z}$$

where $y = D_1^{1/2} e_m$ and $z = D_2^{1/2} e_n$.

Note that condition (3.3) is equivalent to $D_1 \circ P = D_1$ and $D_2 \circ Q = D_2$.

Proof. Let $A \in M_{m,n}$ be given. Suppose that there are matrices $P \in H_m$ and $Q \in H_n$ such that (3.1) holds. Let $X \in M_n$ be such that $\|X\|_\infty \leq 1$. Then

$$\begin{pmatrix} I & X \\ X^* & I \end{pmatrix} \geq 0.$$

Now by the Schur Product Theorem and the fact that $P \circ I \leq I$ and $Q \circ I \leq I$, we have

$$0 \leq \begin{pmatrix} I \circ P & X \circ A \\ X^* \circ A^* & I \circ P \end{pmatrix} \leq \begin{pmatrix} I & A \circ X \\ (A \circ X)^* & I \end{pmatrix}.$$

Hence $\|A \circ X\|_\infty \leq 1$. So $\|A\|_{H,\infty} \leq 1$.

To prove the other direction and the second statement consider

$$c(A) \equiv \max \left\{ \lambda_{\min} \begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} : P \in H_n, Q \in H_m, P \circ I \leq I, \text{ and } Q \circ I \leq I \right\}.$$

The maximum is attained by a compactness argument. Let us first consider the case $c(A) = 0$. By Lemma 1.4 this maximum is 0 and is attained at the matrices P, Q if and only if there are matrices D_1, D_2, Y such that (3.2) and (3.3) hold. The first inequality in (3.2) yields

$$\begin{pmatrix} I & D_1^{1/2\uparrow} \bar{Y} D_2^{1/2\uparrow} \\ (D_1^{1/2\uparrow} \bar{Y} D_2^{1/2\uparrow})^* & I \end{pmatrix} \geq 0$$

so $\|D_1^{1/2\uparrow} \bar{Y} D_2^{1/2\uparrow}\|_\infty \leq 1$. By the second condition in (3.2) and Lemma 1.1, we have

$$e_{2n}^* \begin{pmatrix} D_1 \circ P & A \circ \bar{Y} \\ (A \circ \bar{Y})^* & D_2 \circ Q \end{pmatrix} e_{2n} = 0.$$

The complementarity conditions (3.3) imply $D_1 \circ P = D_1$ and $D_2 \circ Q = D_2$. So

$$e_{2n}^* \begin{pmatrix} D_1 & A \circ \bar{Y} \\ (A \circ \bar{Y})^* & D_2 \end{pmatrix} e_{2n} = 0.$$

Since the matrix in this expression is positive semidefinite, $\|A \circ [D_1^{1/2\uparrow} \bar{Y} D_2^{1/2\uparrow}]\|_\infty = 1$ and, consequently, $\|A\|_{H,\infty} = 1$.

Now suppose that there are no matrices P, Q such that (3.1) holds. Then $c(A) < 0$. The rest of the proof, i.e., showing the ‘only if’ part of (a), is the same as the end of the proof of the previous theorem. $\square$

Theorem 3.2. *Let $A \in M_n$.*

(a) $\|A\|_{H, \omega} \leq 1$ *if and only if there is a matrix $P \in M_n$ such that*

$$(3.4) \quad \begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \geq 0 \quad \text{and} \quad P \circ I \leq I.$$

(b) *Suppose that $P \in H_n$ is such that (3.4) holds. Then $\|A\|_{H, \omega} = 1$ if and only if there are matrices $D_1, D_2, H \in M_n$ and $Y \in M_n$ with D_1 and D_2 diagonal and H Hermitian with main diagonal zero such that*

$$(3.5) \quad \begin{pmatrix} D_1 + H & Y \\ Y^* & D_2 - H \end{pmatrix} \geq 0, \quad \begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \begin{pmatrix} D_1 + H & Y \\ Y^* & D_2 - H \end{pmatrix} = 0,$$

$$(3.6) \quad \Delta(P \circ I - I) = 0$$

(where $\Delta = (D_1 + D_2)/2$). For any such D_1, D_2, H , and Y

$$\omega(\Delta^{\dagger 1/2} \bar{Y} \Delta^{\dagger 1/2}) = 1 \quad \text{and} \quad \omega(A \circ (\Delta^{\dagger 1/2} \bar{Y} \Delta^{\dagger 1/2})) = 1.$$

In particular,

$$y^*(A \circ [\Delta^{\dagger 1/2} \bar{Y} \Delta^{\dagger 1/2}])y = y^* y,$$

where $y = \Delta^{1/2} e_n$.

Proof. Let $A \in M_n$ be given. Suppose that there is a matrix $P \in H_n$ such that (3.4) holds. Let $X \in M_n$ be such that $\omega(X) \leq 1$. Then, by Theorem 2.1, there is $Z \in H_n$ such that

$$\begin{pmatrix} I + Z & X \\ X^* & I - Z \end{pmatrix} \geq 0.$$

Now by the Schur Product Theorem and the fact that $P \circ I \leq I$, we have

$$0 \leq \begin{pmatrix} (I + Z) \circ P & X \circ A \\ X^* \circ A^* & (I - Z) \circ P \end{pmatrix} \leq \begin{pmatrix} I + P \circ Z & A \circ X \\ (A \circ X)^* & I - P \circ Z \end{pmatrix}.$$

Hence, by Theorem 2.1 again, $\omega(A \circ X) \leq 1$. So $\|A\|_{H, \omega} \leq 1$.

To prove the other direction and the second statement consider

$$c(A) \equiv \max \left\{ \lambda_{\min} \begin{pmatrix} P & A \\ A^* & P \end{pmatrix} : P \in H_n \text{ and } P \circ I \leq I \right\}.$$

The maximum is attained by a compactness argument. First consider the case $c(A) = 0$. By Lemma 1.4 this maximum is 0 and is attained at the matrix P if and only if there are matrices D_1, D_2, H, Y such that (3.5) and (3.6) hold. Let $\Delta = (D_1 + D_2)/2$ and $D_3 = (D_1 - D_2)/2$. Then

$$\begin{pmatrix} I + \Delta^{\dagger 1/2} (D_3 + H) \Delta^{\dagger 1/2} & \Delta^{\dagger 1/2} Y \Delta^{\dagger 1/2} \\ (\Delta^{\dagger 1/2} Y \Delta^{\dagger 1/2})^* & I - \Delta^{\dagger 1/2} (D_3 + H) \Delta^{\dagger 1/2} \end{pmatrix} \geq 0$$

so, by Theorem 2.1, $\omega(\Delta^{\dagger 1/2} \bar{Y} \Delta^{\dagger 1/2}) \leq 1$. Also, by the second condition in (3.5) and Lemma 1.1, we have

$$e_{2n}^* \begin{pmatrix} D_1 + P \circ \bar{H} & A \circ \bar{Y} \\ (A \circ \bar{Y})^* & D_2 - P \circ \bar{H} \end{pmatrix} e_{2n} = 0$$

or, equivalently,

$$\begin{pmatrix} \Delta^{1/2} e_n \\ \Delta^{1/2} e_n \end{pmatrix}^* \begin{pmatrix} I + \Delta^{\dagger 1/2} (P \circ \overline{H} + D_3) \Delta^{\dagger 1/2} & A \circ \Delta^{\dagger 1/2} \overline{Y} \Delta^{\dagger 1/2} \\ (A \circ \Delta^{\dagger 1/2} \overline{Y} \Delta^{\dagger 1/2})^* & I - \Delta^{\dagger 1/2} (P \circ \overline{H} + D_3) \Delta^{\dagger 1/2} \end{pmatrix} \begin{pmatrix} \Delta^{1/2} e_n \\ \Delta^{1/2} e_n \end{pmatrix} = 0.$$

By Lemma 2.1 above or a straightforward computation, this implies that $\omega(A \circ [\Delta^{\dagger 1/2} \overline{Y} \Delta^{\dagger 1/2}]) \geq 1$ and that $\|A\|_{H, \omega} = 1$. The rest of the proof is the same as that of Theorem 3.1. $\square$

The rest of this section is devoted to corollaries of Theorems 3.1 and 3.2.

The following result is new. It is natural and follows easily from the constructive proofs given in the previous section. The identity (3.7) verifies Conjecture 6.10 in [9].

Corollary 3.3. *Let $A \in M_{m,n}(R)$ and $B \in M_n(R)$. Then*

$$\|A\|_{H, \infty} = \max\{\|A \circ X\|_{\infty} : \|X\|_{\infty} \leq 1 \text{ and } X \in M_{m,n}(R)\}$$

and

$$\|A\|_{H, \omega} = \max\{\omega(A \circ X) : \omega(X) \leq 1 \text{ and } X \in M_n(R)\}.$$

Let $A \in H_n$. Then

$$(3.7) \quad \|A\|_{H, \infty} = \max\{\|A \circ X\|_{\infty} : \|X\|_{\infty} \leq 1 \text{ and } X \in H_n\}$$

and

$$(3.8) \quad \|A\|_{H, \omega} = \max\{\omega(A \circ X) : \omega(X) \leq 1 \text{ and } X \in H_n\}.$$

Proof. The proofs of the results for A real are very similar to the case where A is Hermitian and are omitted. In the Hermitian case we will prove the result for the norm $\|\cdot\|_{H, \omega}$. The result for $\|\cdot\|_{H, \infty}$ follows from this because $\|A\|_{H, \infty} \leq \|A\|_{H, \omega}$ for any $A \in M_n$. (In fact, it is easy to show from the representations in Theorems 3.1 and 3.2 that $\|A\|_{H, \infty} = \|A\|_{H, \omega}$ if $A = A^*$. See [2, Corollary 3] for the details.)

Let $A \in H_n$ be given. We may assume that $\|A\|_{H, \omega} = 1$. Then by Theorem 3.2 there are matrices D_1, D_2, P, H, Y such that

$$\begin{pmatrix} D_1 + H & Y \\ Y^* & D_2 - H \end{pmatrix} \geq 0, \quad \begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \geq 0$$

and

$$(3.9) \quad \begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \begin{pmatrix} D_1 + H & Y \\ Y^* & D_2 - H \end{pmatrix} = 0.$$

However, because $A = A^*$, applying a block permutation similarity to equation (3.9) gives

$$(3.10) \quad \begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \begin{pmatrix} D_2 - H & Y^* \\ Y & D_1 + H \end{pmatrix} = 0.$$

Adding these two equations gives

$$(3.11) \quad \begin{pmatrix} P & A \\ A^* & p \end{pmatrix} \begin{pmatrix} D_1 + D_2 & Y + Y^* \\ Y + Y^* & D_1 + D_2 \end{pmatrix} = 0.$$

It is clear that

$$\begin{pmatrix} D_1 + D_2 & Y + Y^* \\ Y + Y^* & D_1 + D_2 \end{pmatrix}$$

being the sum of two positive semidefinite matrices is itself positive semidefinite. Thus, by Theorem 2.1,

$$\omega(A \circ [(D_1 + D_2)^{\dagger/2}(Y + Y^*)^T(D_1 + D_2)^{\dagger/2}]) = 1.$$

Since the matrix $(D_1 + D_2)^{\dagger/2}(Y + Y^*)^T(D_1 + D_2)^{\dagger/2}$ is Hermitian, (3.8) follows. $\square$

It is clear from Theorem 3.1 that $\|A\|_{H,\infty} \leq \|A\|_\infty$ for any matrix A . The case of equality was characterized in [11]; we show that the characterization follows easily from Theorem 3.1.

Corollary 3.4. *Let $\|A\|_\infty = 1$. Then $\|A\|_{H,\infty} = 1$ if and only if a submatrix of A is unitary.*

Proof. Let $A \in M_n$ with $\|A\|_\infty = 1$ be given. Then

$$\begin{pmatrix} I & A \\ A^* & I \end{pmatrix} \geq 0.$$

If $\|A\|_{H,\infty} = 1$ then by the second statement in Theorem 3.1 there must be matrices D_1, D_2, Y of the appropriate dimensions such that

$$\begin{pmatrix} D_1 & Y \\ Y^* & D_2 \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} I & A \\ A^* & I \end{pmatrix} \begin{pmatrix} D_1 & Y \\ Y^* & D_2 \end{pmatrix} = 0.$$

Assume, without loss of generality, that the first k_1 (respectively, k_2) main diagonal entries of D_1 (respectively, D_2) are positive and the remaining entries are 0. Let $\widehat{D}_1$ (respectively, $\widehat{D}_2, \widehat{Y}, \widehat{A}$) be the $k_1 \times k_1$ (respectively, $k_2 \times k_2, k_1 \times k_2, k_1 \times k_2$) submatrix in the upper left corner of D_1 (respectively, D_2, Y, A). Then

$$\begin{pmatrix} I_{k_1} & \widehat{A} \\ \widehat{A}^* & I_{k_2} \end{pmatrix} \begin{pmatrix} \widehat{D}_1 & \widehat{Y} \\ \widehat{Y}^* & \widehat{D}_2 \end{pmatrix} = 0.$$

Consideration of the 1, 1 and 2, 1 entries in the product gives

$$\widehat{D}_1 = -\widehat{A}\widehat{Y}^* \quad \text{and} \quad \widehat{Y}^* = -\widehat{A}^*\widehat{D}_1.$$

So $\widehat{D}_1 = \widehat{A}\widehat{A}^*\widehat{D}_1$, and now by the nonsingularity of $\widehat{D}_1$ we have $I_{k_1} = \widehat{A}\widehat{A}^*$. Similarly, $I_{k_2} = \widehat{A}^*\widehat{A}$. Thus $\widehat{A}$ must be unitary. $\square$

4. GENERALIZATIONS AND DUAL NORMS

In this section we generalize the ideas of §§2 and 3 and relate the results to those for dual norms. Given a norm $\|\cdot\|$ on $M_{m,n}$, we define its dual norm by $\|A\|^* = \max\{|\operatorname{tr} AB^*| : \|B\| \leq 1\}$.

Theorem 4.1. *Let m, n be given positive integers. Let S be a closed convex subset of H_{m+n} such that any $P \in S$ may be written $P_1 \oplus P_2$ where $P_1 \in H_m$ and $P_2 \in H_n$. Define a function $N(\cdot)$ on $M_{m,n}$ by*

$$(4.1) \quad N(A) = \inf \left\{ \alpha \geq 0 : \alpha P + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \geq 0 \text{ for some } P \in S \right\}.$$

Then $N(\cdot)$ is a norm if and only if

- (a) *there is a positive definite matrix $P \in S$;*
- (b) *$S_+ = S \cap \{P \geq 0\}$ is compact.*

In this case the infimum in (4.1) is attained for any $A \in M_{m,n}$.

Proof. First we show that if conditions (a) and (b) hold then $N(\cdot)$ is a norm. It is clear that $N(\beta A) = |\beta|N(A)$ and that $N(A + B) \leq N(A) + N(B)$ for any $A, B \in M_{m,n}$ and any $\beta \in C$ and that $N(0) = 0$. So in order to show that $N(\cdot)$ is a norm it suffices to show that $N(A) \in (0, \infty)$ for any nonzero $A \in M_{m,n}$.

Let $A \in M_{m,n}$ be a given nonzero matrix. By condition (a) there is an $\varepsilon > 0$ $P \in S$ such that $P \geq \varepsilon I$. So

$$(\varepsilon^{-1}\|A\|_\infty)P + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \geq (\varepsilon^{-1}\|A\|_\infty)I + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \geq 0,$$

and hence $N(A) \leq \varepsilon^{-1}\|A\|_\infty < \infty$. By condition (b) there is a positive constant c such that $P \leq cI$ for all $P \in S$. For any $\alpha < C^{-1}\|A\|_\infty$ we have

$$\alpha P + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} < \|A\|_\infty I + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}.$$

The right-hand side is not positive definite and hence the left-hand side cannot be positive semidefinite. Thus $N(A) \geq c^{-1}\|A\|_\infty$. We showed that $N(\cdot)$ is a norm. By the second condition the set $\{P \in S: P \geq 0\}$ is a compact set, and hence the infimum is attained for any $A \in M_{m,n}$.

Now we show that if $N(\cdot)$ is a norm then conditions (a) and (b) must be satisfied. First we show that (a) is necessary.

Let $z \in C^{m+n}$ be a given nonzero vector. We will show that there is a matrix $P \in S_+$ such that $z^*Pz > 0$. Write $z = \begin{pmatrix} x \\ y \end{pmatrix}$ with $x \in C^m$ and $y \in C^n$. Without loss of generality we may assume that y is not 0. Because $N(e_my^*) = c < \infty$ there must be a matrix $P = P_1 \oplus P_2 \in S_+$ such that

$$\begin{pmatrix} cP_1 & e_my^* \\ ye_m^* & cP_2 \end{pmatrix} \geq 0.$$

This implies that P_1 is positive semidefinite and that y is in the range of P_2 and hence $y^*P_2y > 0$. So $z^*Pz = x^*P_1x + y^*P_2y \geq y^*P_2y > 0$. Thus we showed that for each unit vector $z \in C^{m+n}$ there is a matrix $P \in S$ such that $z^*Pz > 0$; that is,

$$\nexists \lambda_i > 0, \quad z_i \in C^{m+n} \text{ with } z_i^*z_i = 1 \text{ such that } \sum_{i=1}^k \lambda_i z_i z_i^* \in S^*.$$

By Theorem 1.3, this is equivalent to

$$\exists P \in S \text{ such that } z^*Pz > 0 \text{ for all } z \in C^{m+n}, \quad z^*z = 1.$$

Thus we showed that if $N(\cdot)$ is a norm then S must contain a positive definite matrix.

We now show that if $N(\cdot)$ is a norm then S_+ must be compact. Since S_+ is closed, it is sufficient to show that S_+ is bounded. Let $Q \in S$ be positive definite and let $a = \lambda_{\min}(Q) > 0$. Then $Q \geq aI$. Let $c > 0$ be such that

$$N(A) \geq c\|A\|_\infty \quad \text{for all } A \in M_{m,n}.$$

The existence of such a positive constant is guaranteed by the equivalence of the norms $N(\cdot)$ and $\|\cdot\|_\infty$ on the finite-dimensional vector space $M_{m,n}$ [4,

Corollary 5.4.5]. Let $P \in S_+$ be given and take any $x \in C^m$. Let $y = [1, 0, \dots, 0]^* \in C^n$. Then $N(xy^*) \geq c\|xy^*\|_\infty = c > c/2$. Thus

$$\frac{c}{2} \cdot \frac{Q+P}{2} + \begin{pmatrix} 0 & xy^* \\ yx^* & 0 \end{pmatrix} \not\geq 0.$$

However,

$$\begin{aligned} \frac{c}{2} \cdot \frac{Q+P}{2} + \begin{pmatrix} 0 & xy^* \\ yx^* & 0 \end{pmatrix} &\geq \frac{c}{2} \cdot \frac{aI+P}{2} + \begin{pmatrix} 0 & xy^* \\ yx^* & 0 \end{pmatrix} \\ &\geq \begin{pmatrix} (c/4)P_1 & xy^* \\ yx^* & (ac/4)I \end{pmatrix}, \end{aligned}$$

and hence the last matrix is not positive semidefinite. Since $(c/4)P_1$ is positive semidefinite, we must have $(c/4)x^*P_1x < (4/ac)(x^*x)^2$. Thus $\lambda_{\max}(P_1) < 16/ac$ and, similarly, $\lambda_{\max}(P_2) < 16/ac$. Thus $\lambda_{\max}(P) < 16/ac$ for all $P \in S_+$, which implies that S_+ is bounded. $\square$

Theorem 4.2. Let S be a closed convex subset of H_{m+n} such that any $P \in S$ may be written $P_1 \oplus P_2$ where $P_1 \in H_m$ and $P_2 \in H_n$. Assume that S satisfies the two conditions in Theorem 4.1. Let $N(\cdot)$ be the norm defined in (4.1), and let $N^*(\cdot)$ be its dual. Then for any $A \in M_{m,n}$, $N^*(A) \leq 1$ if and only if there is a matrix $R \in S^D$ such that

$$(4.2) \quad R + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \geq 0,$$

where

$$(4.3) \quad S^D = \{R = R_1 \oplus R_2 : R_1 \in H_m, R_2 \in H_n, \text{ and } \operatorname{tr} RP \leq 2 \forall P \in S\}.$$

Note that the usual definition of S^D would have the condition $\operatorname{tr} RP \leq 1$ instead of $\operatorname{tr} RP \leq 2$.

Proof. We first note that if S satisfies the two conditions in Theorem 4.1 then so does S^D . Also, it is not hard to show that $S^{DD} = S$.

Let $A \in M_{m,n}$ be given. Suppose that there is a matrix $R \in S^D$ such that

$$(4.4) \quad R + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \geq 0.$$

Then for any matrix $B \in M_{m,n}$ with $N(B) \leq 1$, there is a matrix $P \in S$ such that

$$(4.5) \quad P + \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \geq 0.$$

Thus, by (4.4) and (4.5), we have

$$0 \leq \operatorname{tr} \left[R + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \right] \left[P + \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \right] = \operatorname{tr} RP + 2\operatorname{Retr} AB^*,$$

which implies

$$-2\operatorname{Retr}(AB^*) \leq \operatorname{tr} RP \leq 2 \quad \text{for all } B \text{ with } N(B) \leq 1.$$

Thus $N^*(A) \leq 1$.

Let α be the least number such that there is a positive semidefinite matrix $R \in S^D$ such that

$$Z \equiv \alpha R + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \geq 0.$$

Let $\widehat{R} \in S^D$ be such a matrix. We must show that $N(A) \geq \alpha$. By the choice of $\widehat{R}$, Lemma 1.4 guarantees that there is a positive semidefinite nonzero matrix M in the polar or the cone of feasible directions at $\widehat{R}$ such that

$$(4.6) \quad \left[\alpha \widehat{R} + \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \right] M = 0.$$

Let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix}.$$

Because M is in the polar of the cone of feasible directions at $\widehat{R}$,

$$\text{tr} \begin{pmatrix} \widehat{R}_1 - R_1 & 0 \\ 0 & \widehat{R}_2 - R_2 \end{pmatrix} M \leq 0 \quad \text{for all } R_1 \oplus R_2 \in S^D$$

or, equivalently,

$$(4.7) \quad \text{tr}(M_{11} \oplus M_{22})R \leq \text{tr}(M_{11} \oplus M_{22})\widehat{R} \quad \text{for all } R \in S^D.$$

Because there is a positive definite matrix R in S^D , the right-hand side of (4.7) must be positive. Thus, multiplying M by a positive constant if necessary, we may assume that the right-hand side of (4.7) is 2. That is

$$(4.8) \quad \text{tr}(M_{11} \oplus M_{22})R \leq 2 \quad \text{for all } R \in S^D,$$

which says that $M_{11} \oplus M_{22} \in S^D D$, which in turn implies that $N(M_{12}) \leq 1$. Condition (4.6) now yields

$$2 \text{Re tr}(A^* M_{12}) = -\alpha \text{tr}(\widehat{R} M) = -2\alpha.$$

Since $N(M_{12}) \leq 1$, this implies $N^*(A) \geq \alpha$. $\square$

The trace norm is defined on $M_{m,n}$ by $\|A\|_1 = \text{tr}|A|$. It is well known that the trace norm is the dual of the spectral norm. In the following corollary we apply Theorem 4.2 to the norms $\|\cdot\|_\infty$, $\omega(\cdot)$, $\|\cdot\|_{H,\infty}$, and $\|\cdot\|_{H,\omega}$.

Corollary 4.3. *Let $A \in M_{m,n}$ and $B \in M_n$. Then*

(a) $\|A\|_1 \leq 1$ if and only if there are matrices $P \in H_m$ and $Q \in H_n$ such that

$$\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \geq 0, \quad \text{tr } P \geq 1, \quad \text{tr } Q \geq 1.$$

(b) $\omega^*(B) \leq 1$ if and only if there is a matrix $P \in H_n$ such that

$$\begin{pmatrix} P & B \\ B^* & P \end{pmatrix} \geq 0, \quad \text{tr } P \leq 1.$$

(c) $\|A\|_{H,\infty}^* \leq 1$ if and only if there are diagonal matrices $D_1 \in H_m$ and $D_2 \in H_n$ such that

$$\begin{pmatrix} D_1 & A \\ A^* & D_2 \end{pmatrix} \geq 0, \quad \text{tr } D_1 \leq 1, \quad \text{tr } D_2 \leq 1.$$

(d) $\|B\|_{H,\omega}^* \leq 1$ if and only if there is a diagonal matrix $D \in H_n$ and a Hermitian matrix $Z \in H_n$ (with main diagonal not necessarily zero) such that

$$\begin{pmatrix} D+Z & B \\ B^* & D-Z \end{pmatrix} \geq 0, \quad \text{tr } D \leq 1.$$

There are other well-known norms that are of the type described in Theorem 4.1. In particular, the unitarily invariant norms on $M_{m,n}$ are such norms. Let $\|\cdot\|$ be a unitarily invariant norm on $M_{m,n}$. Then it follows from [6, Corollary 3.2] that for any $A \in M_{m,n}$, $\|A\| \leq 1$ if and only if there are matrices $P \in H_m$, $Q \in H_n$ such that

$$\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \geq 0, \quad \|P\| \leq 1, \quad \|Q\| \leq 1.$$

(How to interpret $\|X\|$ in a consistent manner when $X \notin M_{m,n}$ is explained in [5].)

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