THE NUMERICAL INDEX OF NONSELFADJOINT OPERATOR ALGEBRAS

YOSHIHIRO NAKAMURA, KICHI-SUKE SAITO, AND KAZUNARI SAKABA

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let $\mathfrak{A}$ be a (not necessarily selfadjoint) closed subalgebra of $B(H)$ that is the algebra of all bounded linear operators on a Hilbert space H . In this note, we prove that the range of numerical index of $\mathfrak{A}$ as an algebra is the whole of the interval $[\frac{1}{2}, 1]$.

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H . Given $T \in B(H)$, we define the numerical range $W(T)$ and the numerical radius $w(T)$ as

$$W(T) = \{(Tx, x) : x \in H, \|x\| = 1\},$$

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

Let $\mathfrak{A}$ be a closed (not necessarily selfadjoint) subalgebra of $B(H)$. As in [1, 2], we recall the numerical index $n(\mathfrak{A})$ of $\mathfrak{A}$ is defined by

$$n(\mathfrak{A}) = \inf\{w(T) : T \in \mathfrak{A}, \|T\| = 1\}.$$

Since $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$ for every $T \in B(H)$, we have $\frac{1}{2} \leq n(\mathfrak{A}) \leq 1$. As in [3], Crabb, Duncan, and McGregor showed that if $\mathfrak{A}$ is selfadjoint, that is, $\mathfrak{A}$ is a C^* -algebra, then $n(\mathfrak{A}) = 1$ or $\frac{1}{2}$ if $\mathfrak{A}$ is commutative or not commutative, respectively. In this note, we shall study the numerical index of nonselfadjoint operator algebras. As a result, we shall prove that the range of numerical index of nonselfadjoint operator algebras is the whole of the closed interval $[\frac{1}{2}, 1]$. That is, we have the following theorem.

Theorem. *For every real number r such that $\frac{1}{2} \leq r \leq 1$, there exists a Hilbert space H and a (nonselfadjoint) closed subalgebra $\mathfrak{A}$ of $B(H)$ such that $n(\mathfrak{A}) = r$.*

Proof. Let r be a real number such that $\frac{1}{2} \leq r \leq 1$. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and let B be a normal operator on a Hilbert space K such that the spectrum $\sigma(B)$ of B is $\{\lambda \in \mathbb{C} : |\lambda| = r\}$. Put $T = A \oplus B$ on $H = \mathbb{C}^2 \oplus K$. We now consider the closed subalgebra $\mathfrak{A}_T$ generated by T and I , that is, $\mathfrak{A}_T$ is the norm closure

Received by the editors August 1, 1991.

1991 *Mathematics Subject Classification.* Primary 47D25, 46L05; Secondary 47A12.

Key words and phrases. Operator algebra, numerical index.

Research partially supported by a Grant-in-Aid for Scientific Research from the Japanese Ministry of Education.

of $\{p(T) : p \text{ is polynomial}\}$. Since T is the convexoid operator such that $\|T\| = 1$ and $w(T) = r$, we have $n(\mathfrak{A}_T) \leq r$. Therefore, to prove $n(\mathfrak{A}_T) \geq r$, it is sufficient to prove that $w(p(T)) \geq r\|p(T)\|$ for every polynomial p , by the continuity of numerical radius.

For every polynomial p , we now have $w(p(T)) = \max\{w(p(A)), w(p(B))\}$, $\|p(T)\| = \max\{\|p(A)\|, \|p(B)\|\}$, and $w(p(B)) = \|p(B)\| = \max_{|z|=r} |p(z)|$. Putting $p(z) = \sum_{n=0}^m a_n z^n$, it is clear that

$$w(p(A)) = w(a_0 I + a_1 A) = |a_0| + \frac{1}{2}|a_1|$$

and

$$\|p(A)\| = \left\| \begin{bmatrix} a_0 & 0 \\ a_1 & a_0 \end{bmatrix} \right\| = (|a_0|^2 + \frac{1}{4}|a_1|^2)^{1/2} + \frac{1}{2}|a_1|.$$

If $\|p(A)\| \leq \|p(B)\|$ then $w(p(A)) \leq \|p(B)\|$, and so $w(p(T)) = \|p(B)\| = \|p(T)\|$. If $\|p(A)\| > \|p(B)\|$, then we have $w(p(T)) \geq \|p(B)\|$ and $\|p(T)\| = \|p(A)\|$. Since

$$\|p(B)\| = \max_{|z| \leq r} |p(z)| = \max_{|z| \leq 1} |a_0 + a_1 r z + a_2 r^2 z^2 + \cdots + a_m r^m z^m|,$$

by the Carathéodory-Fejér Theorem (cf. [9, 2.5]), we have

$$\|p(B)\| \geq \left\| \begin{bmatrix} a_0 & 0 \\ a_1 r & a_0 \end{bmatrix} \right\| = \left(|a_0|^2 + \frac{r^2}{4}|a_1|^2 \right)^{1/2} + \frac{r}{2}|a_1|.$$

Therefore, we have

$$\begin{aligned} w(p(T)) &\geq \|p(B)\| \geq \left(|a_0|^2 + \frac{r^2}{4}|a_1|^2 \right)^{1/2} + \frac{r}{2}|a_1| \\ &\geq \left(r^2|a_0|^2 + \frac{r^2}{4}|a_1|^2 \right)^{1/2} + \frac{r}{2}|a_1| \\ &\geq r \left(\left(|a_0|^2 + \frac{1}{4}|a_1|^2 \right)^{1/2} + \frac{1}{2}|a_1| \right) \\ &= r\|p(A)\| = r\|p(T)\|. \end{aligned}$$

This implies that $n(\mathfrak{A}_T) = r$. This completes the proof.

Remark 1. We recall that the numerical index of a normed algebra $\mathfrak{A}$ is defined by

$$n(\mathfrak{A}) = \inf\{v(a) : a \in \mathfrak{A}, \|a\| = 1\},$$

where $v(a)$ is the numerical radius of a (cf. [1, 2]). Then $e^{-1} \leq n(\mathfrak{A}) \leq 1$. In this case, the range of the numerical index is the whole of the closed interval $[e^{-1}, 1]$.

Remark 2. Take $T \in B(H)$ and consider the norm closed subalgebra $\mathfrak{A}_T$ of $B(H)$ generated by T and I . Then we have an interest in the class of operators satisfying $n(\mathfrak{A}_T) = r$, in particular, $n(\mathfrak{A}_T) = 1$. As in the proof of Theorem, $n(\mathfrak{A}_T) = 1$ if and only if $\|p(T)\| = w(p(T))$ for every polynomial p . In [4, III], Fujii defined the notion of the class of operators, that is, an operator $T \in B(H)$ is hen-spectroid if $\tilde{\sigma}(T)$ is the spectral set for T , where $\tilde{\sigma}(T)$ is the complement of the unbounded component of the complement of the spectrum $\sigma(T)$ of T . By [4, III, Theorem 13], an operator $T \in B(H)$ satisfies $n(\mathfrak{A}_T) = 1$ if and only if T is hen-spectroid.

REFERENCES

1. F. F. Bonsall and J. Duncan, *Numerical ranges of operators on Normed spaces and of elements of Normed algebras*, Cambridge Univ. Press, London and New York, 1971.
2. ———, *Numerical ranges*. II, Cambridge Univ. Press, London and New York, 1973.
3. M. J. Crabb, J. Duncan, and C. M. McGregor, *Characterizations of commutativity for C^* -algebras*, Glasgow Math. J. **15** (1974), 172–175.
4. M. Fujii, *On some examples of non-normal operators*. I, Proc. Japan Acad. **47** (1971), 458–463; II, **48** (1972), 118–123; III, **48** (1972), 124–129.
5. U. Haagerup and P. de la Harpe, *The numerical radius of a nilpotent on a Hilbert space*, preprint, 1988.
6. P. R. Halmos, *A Hilbert space problem book*, 2nd ed., Springer-Verlag, Berlin and New York, 1982.
7. S. Hildebrandt, *Über der numershen Wertebereich eines operators*, Math. Ann. **163** (1966), 230–247.
8. T. Huruya, *The normed space numerical index of C^* -algebras*, Proc. Amer. Math. Soc. **63** (1977), 289–290.
9. M. Rosenblum and J. Rovnyak, *Hardy classes and operator theory*, Oxford Univ. Press, London and New York, 1985.

(Y. Nakamura) DIVISION OF APPLIED MATHEMATICS, RESEARCH INSTITUTE OF APPLIED ELECTRICITY, HOKKAIDO UNIVERSITY, SAPPORO, 060 JAPAN

(K.-S. Saito and K. Sakaba) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NIIGATA UNIVERSITY, NIIGATA, 950-21 JAPAN

E-mail address: F00745@sinet.ad.jp