ASYMPTOTIC BEHAVIOUR OF UNBOUNDED NONEXPANSIVE SEQUENCES IN BANACH SPACES

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ABSTRACT. Let X be a real Banach space, $(x_n)_{n \geq 0}$ a nonexpansive sequence in X (i.e., $\|x_{i+1} - x_{j+1}\| \leq \|x_i - x_j\|$ for all $i, j \geq 0$), and C the closed convex hull of the sequence $(x_{n+1} - x_n)_{n \geq 0}$.

We prove that $\lim_{n \rightarrow +\infty} \|x_n/n\| = \inf_{n \geq 1} \|(x_n - x_0)/n\| = \inf_{z \in C} \|z\|$ and deduce a simple short proof for the following result. (i) If X is reflexive and strictly convex, then x_n/n converges weakly in X to the element of minimum norm $P_C 0$ in C with

$$\|P_C 0\| = \inf_{n \geq 1} \left\| \frac{x_n - x_0}{n} \right\| = \lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\|.$$

(ii) If X^* has Fréchet differentiable norm, then x_n/n converges strongly to $P_C 0$. This result contains previous results by Pazy, Kohlberg and Neyman, Plant and Reich, and Reich and is also optimal since the assumptions made on X in (i) or (ii) are also necessary for the respective conclusion to hold.

1. INTRODUCTION

Let X be a real Banach space; the norms of both X and X^* are denoted by $\|\cdot\|$. We denote weak convergence in X by $\rightharpoonup$ and strong convergence by $\rightarrow$. For notions concerning the geometry of Banach spaces, we refer to [1, 10, 11] without recalling the basic definitions.

Let C be a subset of X ; a mapping $T: C \rightarrow C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. When X is a Hilbert space, C is closed and convex, and $x \in C$, the strong convergence of the sequence $T^n x/n$ to the unique element of minimum norm in $\overline{R(T - I)}$ was first proved by Pazy [15]; subsequently his result was extended to more general Banach spaces and to nonlinear contraction semigroups by many authors; with additional restrictions on the set C , Reich [18, 19] extended Pazy's result to Banach spaces whose norm is uniformly Gâteaux differentiable and whose dual has Fréchet differentiable norm; Kohlberg and Neyman [12] gave a simple geometric proof of Pazy's result in uniformly convex spaces, and in [13] they extended their previous result to give a characterization of Banach spaces for which weak or

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strong convergence of $T^n x/n$ occurs for all nonempty closed convex subsets C of X and all nonexpansive maps $T: C \rightarrow C$; namely, that X is reflexive and strictly convex in the first case and X^* has Fréchet differentiable norm in the latter. By using an idea of [12], Reich [20] was then able to drop the previously mentioned restrictions on the set C , and subsequently Plant and Reich [17] extended Kohlberg and Neyman's result [13] to nonexpansive maps T defined on closed subsets of X , where $(I - T)$ need not satisfy any range condition.

In this paper we consider a nonexpansive map T from an arbitrary subset C of X into itself, and for $x \in C$, we study the asymptotic behavior of the sequence $T^n x/n$ in X . This is equivalent to the study of the asymptotic behaviour of the sequence x_n/n where $(x_n)_{n \geq 0}$ is a sequence in X satisfying $\|x_{i+1} - x_{j+1}\| \leq \|x_i - x_j\|$ for all $i, j \geq 0$, which we shall call a nonexpansive sequence in X .

When X is the real line, this was treated in [6] and when X is a Hilbert space in [7] and subsequently in [8] with a much simpler proof. Our aim here is to extend this to a Banach space, and thus to give a simple proof to extensions of the results previously mentioned.

2. PRELIMINARIES

(a) The sequence $(x_n)_{n \geq 0}$ in X is nonexpansive if $\|x_{i+1} - x_{j+1}\| \leq \|x_i - x_j\|$ for all $i, j \geq 0$.

(b) If A is a subset of X , we denote by $\text{clco } A$ the closed convex hull of A in X and for $x \in X$ $d(x, A) = \inf_{z \in A} \|x - z\|$.

(c) The duality map J from X into the family of nonempty closed convex subsets of X^* is defined by: $\forall x \in X$,

$$J(x) = \{x^* \in X^* / (x^*, x) = \|x\|^2 = \|x^*\|^2\}$$

where (x^*, x) denotes the value of x^* at x . Note also that we have: $\forall x, y \in X, \forall j \in Jx$,

$$(j, x - y) = \|x\|^2 - (j, y) \geq \|x\|^2 - \frac{1}{2}(\|j\|^2 + \|y\|^2) = \frac{1}{2}(\|x\|^2 - \|y\|^2).$$

(d) We recall that if X is reflexive and strictly convex and K a nonempty closed convex subset of X , the nearest point projection map P_K of X onto K is well defined, i.e., K is a Chebyshev set (see [11]).

(e) The norm of X is Fréchet differentiable if for each $x \in S = \{z \in X / \|z\| = 1\}$, $\lim_{t \rightarrow 0} (\|x + ty\| - \|x\|)/t$ exists uniformly for $y \in S$. We recall also that the dual space X^* has Fréchet differentiable norm if and only if X is reflexive, strictly convex, and satisfies the following property: if $x_n \rightarrow_{n \rightarrow \infty} x$ and $\|x_n\| \rightarrow_{n \rightarrow \infty} \|x\|$ then $\|x_n - x\| \rightarrow_{n \rightarrow \infty} 0$ (see [10]).

3. ASYMPTOTIC BEHAVIOUR OF UNBOUNDED NONEXPANSIVE SEQUENCES

When X is a Hilbert space, we studied previously the asymptotic behaviour of bounded nonexpansive sequences in [2, 3] and applied our results to quasi-automonomous dissipative systems in [4, 5]; see also [11, 14, 16] and the references therein for related results.

In this paper we study the asymptotic behaviour of unbounded (or better not necessarily bounded) nonexpansive sequences in a Banach space X ; our main result is the following theorem and its corollary containing previous results by

Pazy [15], Reich [18–20], Kohlberg and Neyman [12, 13], and Plant and Reich [17].

Theorem 3.1. *Let $(x_n)_{n \geq 0}$ be a nonexpansive sequence in X and*

$$C = \text{clco}\{(x_{i+1} - x_i)_{i \geq 0}\}.$$

Then

$$\lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\| = \inf_{n \geq 1} \left\| \frac{x_n - x_0}{n} \right\| = d(O, C).$$

Corollary 3.2. *With the same assumptions as in Theorem 3.1 we have*

(i) *If X is reflexive and strictly convex, then*

$$\frac{x_n}{n} \xrightarrow{n \rightarrow +\infty} P_C O \quad \text{with } \|P_C O\| = \inf_{n \geq 1} \left\| \frac{x_n - x_0}{n} \right\| = \lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\|.$$

(ii) *If X^* has Fréchet differentiable norm, then $x_n/n \rightarrow_{n \rightarrow +\infty} P_C O$.*

Remark 3.3. This result is optimal since, by [13], the assumptions made on X in (i) or (ii) are also necessary for the respective conclusion to hold.

For the proof we need the next proposition, which follows immediately from the following well-known lemma in classical analysis, a proof of which is included for completeness.

Lemma 3.4. *Let $(a_n)_{n \geq 1}$ be a sequence of nonnegative real numbers satisfying $\forall n, p \geq 1, a_{n+p} \leq a_n + a_p$. Then a_n/n converges as $n \rightarrow +\infty$ and $\lim_{n \rightarrow +\infty} a_n/n = \inf_{n \geq 1} a_n/n$.*

Proof. Let $p \geq 1$ be fixed; then: $\forall n \geq p \exists! k \geq 1 \exists! i, 0 \leq i < p$, such that $n = kp + i$.

Hence we have

$$\frac{a_n}{n} = \frac{a_{kp+i}}{kp+i} \leq \frac{a_{kp} + a_i}{kp+i} \leq \frac{a_{kp}}{kp} + \frac{a_i}{kp} \leq \frac{a_p}{p} + \frac{a_i}{kp} \leq \frac{a_p}{p} + \frac{\max_{0 \leq i < p} a_i}{kp}.$$

Now letting $n \rightarrow +\infty$, we have $k \rightarrow +\infty$, thus $\forall p \geq 1, \limsup_{n \rightarrow +\infty} a_n/n \leq a_p/p$. Therefore $\limsup_{n \rightarrow +\infty} a_n/n \leq \inf_{p \geq 1} a_p/p \leq \liminf_{n \rightarrow +\infty} a_n/n$.

Hence $\lim_{n \rightarrow +\infty} a_n/n = \inf_{n \geq 1} a_n/n$.

Proposition 3.5. *We have*

$$\lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\| = \inf_{n \geq 1} \left\| \frac{x_n - x_0}{n} \right\|.$$

Proof. Let $a_n = \|x_n - x_0\|$ for $n \geq 1$. Then we have $a_{n+m} = \|x_{n+m} - x_0\| \leq \|x_{n+m} - x_n\| + \|x_n - x_0\| \leq a_m + a_n$ for $m, n \geq 1$. Hence the result follows from Lemma 3.4.

Proof of Theorem 3.1. Let $k \geq 1$ fixed, and $j_n \in J(x_n - x_{k-1})$ for $n \geq k$. Then we have $\forall n \geq k \geq 1$,

$$\begin{aligned} (j_n, x_k - x_{k-1}) &\geq \frac{1}{2} \|x_n - x_{k-1}\|^2 - \frac{1}{2} \|x_n - x_k\|^2 \\ &\geq \frac{1}{2} \|x_n - x_{k-1}\|^2 - \frac{1}{2} \|x_{n-1} - x_{k-1}\|^2. \end{aligned}$$

Hence

$$(3.1) \quad \frac{2}{n^2} \left(\sum_{i=k}^n j_i, x_k - x_{k-1} \right) \geq \left\| \frac{x_n - x_{k-1}}{n} \right\|^2.$$

Let $A_n = (2/n^2) \sum_{i=k}^n j_i$ for $n \geq k$; we have $\|A_n\| \leq (2/n^2) \sum_{i=k}^n \|x_i - x_{k-1}\|$; hence $\|A_n\|$ is bounded since $\|x_n/n\|$ is bounded. Thus, from the weak star compactness of the closed unit ball of X^* , it follows that the sequence A_n has a weak star cluster point $q^* \in X^*$ (obviously independent of $k \geq 1$). Now by Proposition 3.5 and (3.1) we get: $\forall k \geq 1$,

$$(q^*, x_k - x_{k-1}) \geq \lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\|^2.$$

But we have

$$\begin{aligned} \|q^*\| &\leq \limsup_{n \rightarrow +\infty} \|A_n\| \leq \limsup_{n \rightarrow +\infty} \frac{2}{n^2} \sum_{i=k}^n \|x_i - x_{k-1}\| \\ &= \limsup_{n \rightarrow +\infty} \frac{2}{n^2} \sum_{i=k}^n i \left\| \frac{x_i - x_{k-1}}{i} \right\| = \lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\|. \end{aligned}$$

Therefore, we get: $\forall k \geq 1$,

$$(q^*, x_k - x_{k-1}) \geq \lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\|^2 \geq \|q^*\|^2.$$

Hence $\forall z \in C$,

$$\begin{aligned} (3.2) \quad \frac{1}{2} \lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\|^2 + \frac{1}{2} \|z\|^2 &\geq \frac{1}{2} \|q^*\|^2 + \frac{1}{2} \|z\|^2 \\ &\geq (q^*, z) \geq \lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\|^2 \geq \|q^*\|^2. \end{aligned}$$

Now (3.2) implies that $\|q^*\| \leq \lim_{n \rightarrow +\infty} \|x_n/n\| \leq \inf_{z \in C} \|z\| = d(O, C)$ and, therefore, since $z = (x_n - x_0)/n \in C$, we must have

$$\|q^*\| = \lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\| = d(O, C) = \inf_{n \geq 1} \left\| \frac{x_n - x_0}{n} \right\|.$$

This completes the proof of the theorem.

Proof of Corollary 3.2. X being reflexive, x_n/n has a weakly convergent subsequence; let $x_{n_i}/n_i \rightarrow_{i \rightarrow +\infty} p \in C$; X being also strictly convex, we have $d(O, C) = \|P_C 0\|$. Therefore

$$\|p\| \leq \liminf_{i \rightarrow +\infty} \left\| \frac{x_{n_i}}{n_i} \right\| = \lim_{n \rightarrow +\infty} \left\| \frac{x_n}{n} \right\| = \|P_C 0\|$$

by Theorem 3.1; hence we must have $p = P_C 0$; this shows that $x_n/n \rightarrow_{n \rightarrow +\infty} P_C 0$ with $\|P_C 0\| = \lim_{n \rightarrow +\infty} \|x_n/n\| = \inf_{n \geq 1} \|(x_n - x_0)/n\|$ and completes the proof of (i).

Now (ii) is an immediate consequence of (i) and the characterization of X given in §2(e).

Remark 3.6. Let $K = \bigcap_{n=1}^{\infty} \text{clco}\{(x_i - x_{i-1})_{i \geq n}\}$; then $K \neq \emptyset$ if X is reflexive and in this case the proof of Theorem 3.1 shows that we also have $\lim_{n \rightarrow +\infty} \|x_n/n\| = d(O, K)$; therefore if X is also strictly convex, then $P_C 0 = P_K 0$ since $K \subset C$.

Remark 3.7. Our result may also be applied to the asymptotic behaviour of curves in X , and thus to the asymptotic behaviour of unbounded trajectories

for the quasi-autonomous dissipative system $du/dt + Au \ni f$ where A is an accretive (possibly multivalued) operator in $X \times X$; see [9] for the case X is a Hilbert space.

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