THE PRODUCT OF STRONG OPERATOR MEASURABLE FUNCTIONS IS STRONG OPERATOR MEASURABLE

G. W. JOHNSON

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let $f_1, \dots, f_n$ be strong operator measurable functions with values in the space of bounded linear operators on a separable Hilbert space. We show that the product $f_1 \cdots f_n$ is also strong operator measurable.

1. INTRODUCTION

We begin by describing precisely the setting for our main result. Let X be a Hausdorff topological space and let $\mathcal{B}(X)$ denote the Borel class of X , that is, the σ -field generated by the open subsets of X . μ will be a finite Radon measure (see the definition below as in [11, p. 13]) on $\mathcal{B}(X)$ and $\mathcal{S} = \mathcal{S}_\mu$ will be the σ -field obtained when the measure space $(X, \mathcal{B}(X), \mu)$ is completed. H will be an infinite-dimensional, separable Hilbert space over the complex numbers $\mathbb{C}$ and $\mathcal{L}(H)$ will denote the bounded linear operators on H . We will show that if $f_1, \dots, f_n$ are strong operator measurable functions (definition below) from X to $\mathcal{L}(H)$, then so is the product function $f_1 \cdots f_n$.

It may seem strange to the reader, as it did at first to the writer, that the result that we just stated is not well known. We will make remarks related to this issue at the end of this section.

The definition of “strong operator measurability” of an $\mathcal{L}(H)$ -valued function is given in terms of the “strong measurability” of an H -valued function. Thus we turn to a brief discussion of this latter concept. We take the definition from Hille and Phillips [9, p. 72] because [9] is one of the standard references and because it was the definition used in earlier related work in which the author was involved.

Note. The material in this paper through Theorem 2 is valid with $(X, \mathcal{S}, \mu)$ being a finite, complete measure space, and so we work in this more general setting for now.

Definition 1. A function $g: X \rightarrow H$ is *strongly measurable* (or μ -strongly measurable) if and only if there exists a sequence of functions $\{g_n\}$ such that

Received by the editors August 1, 1991.

1991 *Mathematics Subject Classification.* Primary 46G10, 46G12; Secondary 28B05, 28C15.

Key words and phrases. Strong operator measurability, strong operator continuity, Souslin space, Radon measure, Lusin μ -measurability.

$\|g_n(x) - g(x)\| \rightarrow 0$ for μ -a.e. x and each g_n takes only countably many values, each value being assumed on a set in $\mathcal{S}$.

Hille and Phillips [9, p. 72] note that when the measure space is finite, then the countably-valued functions g_n in Definition 1 can be replaced by finitely-valued functions. This makes it easy to see that for a finite complete measure space the above definition of strong measurability agrees with the definition in Reed and Simon [10, p. 116] and hence will allow us to use a theorem from [10]. Before stating that theorem, we review the definition of “weak measurability”.

Definition 2. A function $g: X \rightarrow H$ is *weakly measurable* (or μ -weakly measurable) if and only if $(g(\cdot), h)$ is a measurable $\mathbb{C}$ -valued function for every $h \in H$ where $(\cdot, \cdot)$ denotes the inner product in H .

Theorem 1 [10, Theorem IV.22, p. 116]. *Let g be a function from the finite, complete measure space $(X, \mathcal{S}, \mu)$ to the separable Hilbert space H . Then the following are equivalent:*

- (a) g is strongly measurable.
- (b) g is $\mathcal{S} - \mathcal{B}(H)$ measurable.
- (c) g is weakly measurable.

Remark 1. (i) Reed and Simon refer to $\mathcal{S} - \mathcal{B}(H)$ measurability as “Borel measurability”. This can be misleading. For example, if $H = \mathbb{C}$, $X = [0, 1]$, $\mu =$ Lebesgue measure, and $\mathcal{S}$ is the class of Lebesgue measurable subsets of $[0, 1]$, then $\mathcal{S} - \mathcal{B}(H)$ measurability of a function g is almost universally called “Lebesgue measurability”. In contrast, Borel measurability means that $g^{-1}(B) \in \mathcal{B}([0, 1])$ for every $B \in \mathcal{B}(\mathbb{C})$.

(ii) Note that the reference to [10] is to the revised and enlarged edition. In the earlier edition the proof of Theorem 1 is not correct since it depends for one thing on part (a) of a preceding proposition whose proof is not correct. The proof in the new edition is correct except for a minor misprint.

We are now ready to define “strong operator measurability” and “weak operator measurability”. As above, the measure space $(X, \mathcal{S}, \mu)$ is taken to be finite and complete. Let f be a function from X to $\mathcal{L}(H)$.

Definition 3. (i) f is said to be *strong operator measurable* if and only if $f(\cdot)h: X \rightarrow H$ is strongly measurable for every $h \in H$.

(ii) f is said to be *weak operator measurable* if and only if $f(\cdot)h: X \rightarrow H$ is weakly measurable for every $h \in H$; that is, $(f(\cdot)h, k)$ is a measurable $\mathbb{C}$ -valued function for every h, k in H .

What we have called strong operator measurability is often referred to simply as strong measurability. However, the use of distinct words for the measurability in Definitions 1 and 3(i) will contribute to clarity as we continue.

How is it possible that the main result in this paper is not well known? We have no clearcut answer, but we finish this introduction with related remarks. These remarks should provide a helpful perspective for some readers, but others may wish to go over them quickly.

Remark 2. (i) The use of the Bochner integral [9] for $\mathcal{L}(H)$ (or H)-valued functions requires consideration of strong operator (or strong) measurability. However, in many situations a weak integral [10, p. 119] is adequate, and strong measurability need never be considered.

(ii) The Bochner integral has certain extremely useful properties [9, Theorems 3.7.4, 3.7.9, 3.7.13, pp. 80–84], which often does make its use essential. This has occurred and has sometimes been explicitly pointed out in the work of the author and his coauthors on or related to the “Feynman integral” (for example, [4, p. 1325, especially Remark 1; 6, pp. 137, 146; 7, pp. 109–110; 8, pp. 368–372, especially Remark 1]). Even when the use of the Bochner integral is crucial and products appear, the necessary strong operator measurability can often be argued directly in specific cases, say when $H = L^2(\mathbb{R}^p)$ [6, pp. 135–136; 7, pp. 107–108]. Such a direct argument was not adequate in a paper that is now in preparation and that led to the present considerations. This new paper is attempting to use the heuristic ideas of Feynman’s operational calculus [3] to develop perturbation series that may be applicable in a variety of situations. Since generality is specifically desired, H is an abstract Hilbert space.

We should mention that if we had known the main result of this paper, the earlier direct arguments yielding the strong operator measurability of products would not have been needed and the proofs of those theorems could have been simplified.

(iii) While many problems about strong operator measurability can be reduced to questions about weak operator measurability via the equivalence of (a) and (c) in Theorem 1, it is not at all clear that this will work with our problem about products. If f_1 and f_2 are weak operator measurable, then for every h, k in H , $(f_1(\cdot)h, k)$, $(f_2(\cdot)h, k)$, and, hence, $(f_1(\cdot)h, k)(f_2(\cdot)h, k)$, are certainly measurable. However, it is the measurability of $(f_1(\cdot)f_2(\cdot)h, k)$ that must be examined in connection with the weak operator measurability of $f_1 \cdot f_2$. Another clue that there could be difficulties here is the fact that multiplication of operators is not even sequentially continuous in the weak operator topology [2, p. 512]. (Measurability and continuity are rather far apart of course, but they are related; see Definition 7 and Theorems 3 and 4.)

Operator multiplication is not continuous in the strong operator topology either [10, p. 183], but it is sequentially continuous [10, p. 216], and even continuous when the factors are restricted to norm bounded subsets of $\mathcal{L}(H)$ [2, p. 512]. A central idea in the proof of our main theorem is to use the restricted continuity just mentioned and to involve the concept of “Lusin μ -measurability” where μ is a Radon measure. The author would like to thank Dr. Brian Jefferies, University of New South Wales, who pointed out the possible relevance of Lusin μ -measurability.

2. STRONG OPERATOR MEASURABLE FUNCTIONS AND THEIR PRODUCTS

We will prove a theorem characterizing strong operator measurability in terms of σ -fields in the usual measure theoretic way. We begin by reviewing the standard basic open sets for the strong operator topology on $\mathcal{L}(H)$.

Let $A_0 \in \mathcal{L}(H)$. Given $\varepsilon > 0$ and any finite subset, say $h_1, \dots, h_n$ of H , we let

$$U(A_0; h_1, \dots, h_n; \varepsilon) := \{A \in \mathcal{L}(H) : \|Ah_i - A_0h_i\| < \varepsilon \text{ for } i = 1, \dots, n\}.$$

The collection of all sets of the above form where ε ranges over the positive real numbers, n ranges over the positive integers, and $h_1, \dots, h_n$ are chosen from H provides a basis for the neighborhood system for A_0 ; the sets of this

form with $n = 1$ give a subbasis. By letting A_0 range over $\mathcal{L}(H)$, a basis (or subbasis, if $n = 1$) for the strong operator topology is obtained.

Definition 4. Let $\mathcal{B}_0 = \mathcal{B}_0(\mathcal{L}(H))$ be the σ -field of subsets of $\mathcal{L}(H)$ generated by the sets in the basis for the strong operator topology.

We will see later on that $\mathcal{B}_0(\mathcal{L}(H)) = \mathcal{B}(\mathcal{L}(H))$, the σ -field generated by the strong operator open subsets of $\mathcal{L}(H)$. Once we know that, we will be able to improve on our next result.

Proposition 1. Let $(X, \mathcal{S}, \mu)$ be a finite complete measure space. Then $f: X \rightarrow \mathcal{L}(H)$ is strong operator measurable if and only if f is $\mathcal{S} - \mathcal{B}_0(\mathcal{L}(H))$ measurable.

Proof. We assume first that f is strong operator measurable. If we show that $f^{-1}(U)$ is $\mathcal{S}$ -measurable for an arbitrary subbasic open set $U = U(A_0; h_0; \varepsilon)$, it will follow that f is $\mathcal{S} - \mathcal{B}_0(\mathcal{L}(H))$ measurable. But

$$\begin{aligned} f^{-1}(U) &= \{x \in X : \|f(x)h_0 - A_0h_0\| < \varepsilon\} \\ &= f_{h_0}^{-1}(\{h \in H : \|h - A_0h_0\| < \varepsilon\}) \end{aligned}$$

where $f_{h_0}(x) := f(x)h_0$. Since (a) implies (b) in Theorem 1, we now see that $f^{-1}(U) \in \mathcal{S}$.

Conversely, we assume that f is $\mathcal{S} - \mathcal{B}_0(\mathcal{L}(H))$ measurable. Because (b) implies (a) in Theorem 1, it suffices to show that for every $h_0, h_1 \in H$ and $\varepsilon > 0$, $f_{h_0}^{-1}(B_\varepsilon(h_1))$ is in $\mathcal{S}$ where f_{h_0} is defined as in the first part of the proof and $B_\varepsilon(h_1)$ denotes the ε -ball around h_1 . Note that

$$f_{h_0}^{-1}(B_\varepsilon(h_1)) = \{x \in X : \|f(x)h_0 - h_1\| < \varepsilon\}.$$

Now take $A_0 \in \mathcal{L}(H)$ such that $A_0h_0 = h_1$. The rank one operator

$$A_0h := \frac{(h, h_0)}{\|h_0\|^2} h_1$$

is one such operator. (The case $h_0 = 0$ requires separate consideration but is trivial.) Now $U(A_0; h_0; \varepsilon) = \{A \in \mathcal{L}(H) : \|Ah_0 - h_1\| < \varepsilon\}$. By our present assumption, $f^{-1}(U(A_0; h_0; \varepsilon))$ is in $\mathcal{S}$. However,

$$f^{-1}(U(A_0; h_0; \varepsilon)) = \{x \in X : \|f(x)h_0 - h_1\| < \varepsilon\} = f_{h_0}^{-1}(B_\varepsilon(h_1)).$$

Thus $f_{h_0}^{-1}(B_\varepsilon(h_1)) \in \mathcal{S}$ as we wished to show. $\square$

Certain facts about “Souslin spaces” will be involved in the remainder of this paper.

Definition 5. Let Z be a Hausdorff topological space.

- (i) Z is a *Polish space* if and only if it is separable and it can be metrized by means of a complete metric.
- (ii) Z is *Lusin* if and only if it is the image of a Polish space under a continuous bijection.
- (iii) Z is *Souslin* if and only if it is the image of a Polish space under a continuous surjection.

It will be convenient to formally state the following trivial fact.

Proposition 2. *Every Polish space is a Lusin space and every Lusin space is a Souslin space.*

There are important spaces that are nonmetrizable Lusin spaces. For example [1, pp. 292–293], if X is any separable, infinite-dimensional Banach space, then X with its weak topology is nonmetrizable and Lusin. If, in addition, the dual space X^* of X is separable, then X^* is nonmetrizable and Lusin in its weak* topology. The example that is most relevant to our present concerns is identified in our next proposition.

Proposition 3 [11, Theorem 7, pp. 112–114]. *Let H be a separable Hilbert space over $\mathbb{C}$. Then $\mathcal{L}(H)$ with its strong operator topology is a Lusin space.*

The following proposition about Souslin spaces will enable us to resolve the issue that was raised just before Proposition 1. This result is taken from [1, Exercise 8, p. 295]. It can be better appreciated if one realizes that a Souslin space need not be 1st countable much less 2nd countable.

Proposition 4. *Let Y be a Souslin space. If $\mathcal{U}$ is any collection of open subsets of Y , then there is a countable subcollection $\mathcal{U}_0$ of $\mathcal{U}$ having the same union.*

Corollary 1. *Let H be a separable Hilbert space over $\mathbb{C}$. The strong operator subbasic open subsets are a generating class for $\mathcal{B}(\mathcal{L}(H))$ where $\mathcal{B}(\mathcal{L}(H))$ is the Borel class of $\mathcal{L}(H)$ equipped with the strong operator topology. Thus $\mathcal{B}_0(\mathcal{L}(H)) = \mathcal{B}(\mathcal{L}(H))$.*

Proof. Clearly $\mathcal{B}_0 \subset \mathcal{B}$. Since every basic open subset is a finite intersection of subbasic open sets, it suffices to show that every nonempty strong operator open set O is the countable union of basic open sets. The set O is, by the nature of a basis, the union of a family $\mathcal{U}$ of basic open sets. By Propositions 2, 3, and 4 there is a countable subcollection $\mathcal{U}_0$ of $\mathcal{U}$ whose union is the set O . $\square$

Proposition 1 and Corollary 1 immediately yield our next result, which has its own independent interest and was not previously known to this writer.

Theorem 2. *Let $(X, \mathcal{S}, \mu)$ be a finite complete measure space, and let H be a separable Hilbert space over $\mathbb{C}$. Then $f: X \rightarrow \mathcal{L}(H)$ is strong operator measurable if and only if f is $\mathcal{S} - \mathcal{B}(\mathcal{L}(H))$ measurable where $\mathcal{L}(H)$ is topologized with the strong operator topology. Further, to establish these two equivalent forms of measurability, it suffices to show that $f^{-1}(U) \in \mathcal{S}$ where $U = U(A_0; h_0; \varepsilon)$ is an arbitrary subbasic open subset for the strong operator topology.*

For the rest of this paper, we will require that X be a Hausdorff topological space and that μ be a “Radon measure” on X .

Definition 6. Let X be a Hausdorff topological space and let μ be a measure on $\mathcal{B}(X)$, the Borel class of X . μ is a Radon measure on X if and only if the following three conditions hold:

- (i) μ is locally finite; i.e., for every $x \in X$ there is an open set O such that $x \in O$ and $\mu(O) < \infty$.
- (ii) μ is outer regular; i.e., for every $B \in \mathcal{B}(X)$,

$$\mu(B) = \inf\{\mu(O) : B \subset O \text{ and } O \text{ is open}\}.$$

(iii) For every open set O ,

$$\mu(O) = \sup\{\mu(K) : K \subset O \text{ and } K \text{ is compact}\}.$$

We will not need the following result, but it assures us that there is a rich supply of finite Radon measures.

Proposition 5 [1, Theorem 8.6.13, p. 294]. *Every finite Borel measure on a Souslin space is a Radon measure.*

Remark 3. The rest of this paper depends on the theory of Radon measures, which for a long time was limited to locally compact topological spaces. However, the locally compact setting is inadequate for many problems of modern analysis, and so a more general theory of Radon measures has been developed. Indeed, the book of Schwartz [11], which is frequently referred to in this paper, is a major contribution to this subject.

Next we define “Lusin μ -measurability”.

Definition 7. Let X and Y be Hausdorff topological spaces and let μ be a Radon measure on X . A function $f: X \rightarrow Y$ is said to be *Lusin μ -measurable* if and only if for every compact subset K of X and for every $\delta > 0$ there exists a compact set K_δ with $K_\delta \subset K$, $\mu(K \setminus K_\delta) < \delta$, and the restriction $f|_{K_\delta}$ of f to K_δ continuous.

Now we turn to two results from [11] that relate Lusin μ -measurability to measurability as usual defined in terms of σ -fields.

Theorem 3. *Let X and Y be Hausdorff topological spaces and let μ be a Radon measure on X . Further, let $\mathcal{S} = \mathcal{S}_\mu$ be the σ -field obtained when $\mathcal{B}(X)$ is completed with respect to μ . Let $f: X \rightarrow Y$.*

- (i) [11, Theorem 5, p. 26] *If f is Lusin μ -measurable then f is $\mathcal{S}_\mu - \mathcal{B}(Y)$ measurable.*
- (ii) [11, Theorem 14, p. 129] *Let Y be a Souslin space. If f is $\mathcal{S}_\mu - \mathcal{B}(Y)$ measurable then f is Lusin μ -measurable.*

Using Theorem 3 and Proposition 3, we can now strengthen Theorem 2 under the additional assumption that X is a Hausdorff topological space and μ is a Radon measure on X .

Theorem 4. *We suppose that X is a Hausdorff topological space and that μ is a finite Radon measure on X . Let $\mathcal{S} = \mathcal{S}_\mu$ be the σ -field obtained when $(X, \mathcal{B}(X), \mu)$ is completed. Finally, let H be a separable Hilbert space over $\mathbb{C}$ and take $f: X \rightarrow \mathcal{L}(H)$. Then the following are equivalent:*

- (i) *f is strong operator measurable.*
- (ii) *f is $\mathcal{S}_\mu - \mathcal{B}(\mathcal{L}(H))$ measurable when $\mathcal{L}(H)$ is equipped with the strong operator topology.*
- (iii) *f is Lusin μ -measurable.*

Further, to establish all three types of measurability above, it suffices to show that $f^{-1}(U) \in \mathcal{S}_\mu$ for an arbitrary subbasic open subset U for the strong operator topology on $\mathcal{L}(H)$.

We are now prepared to show that the product of strong operator measurable functions is strong operator measurable.

Theorem 5. *Let X be a Hausdorff topological space and suppose that μ is a finite Radon measure on X . Let $\mathcal{S} = \mathcal{S}_\mu$ be the σ -field obtained when $(X, \mathcal{B}(X), \mu)$ is completed. Finally, let H be a separable Hilbert space over $\mathbb{C}$.*

If $f_i: X \rightarrow \mathcal{L}(H)$ is strong operator measurable for $i = 1, \dots, n$, then the product function $f_1 \cdots f_n$ is also strong operator measurable.

Proof. A simple induction argument shows that it is enough to deal with the case $n = 2$. By Theorem 4, both f_1 and f_2 are Lusin μ -measurable, and it suffices to show that $f_1 \cdot f_2$ is Lusin μ -measurable. Accordingly, let a compact subset K of X and a $\delta > 0$ be given. We seek a compact subset K_δ such that $K_\delta \subset K$, $\mu(K \setminus K_\delta) < \delta$, and the restriction $(f_1 \cdot f_2)_{K_\delta}$ of $f_1 \cdot f_2$ to K_δ is continuous. Since each f_i is Lusin μ -measurable, there exist compact sets K_i ($i = 1, 2$) such that $K_i \subset K$, $\mu(K \setminus K_i) < \delta/2$, and $(f_i)_{K_i}$ is strong operator continuous. Since K_i is compact in its own topology and the continuous image of a compact set is compact, we see that $f_i(K_i)$ is compact in the strong operator topology on $\mathcal{L}(H)$. But $f_i(K_i)$ compact in the strong operator topology implies that for every $h \in H$, $f_i(K_i)h = \{f_i(x)h : x \in K_i\}$ is compact in H and thus bounded in H . Hence, by the Uniform Boundedness Theorem, $f_i(K_i)$ is an operator norm bounded subset of $\mathcal{L}(H)$.

Now let $K_\delta := K_1 \cap K_2$. Of course, K_δ is compact and is contained in K_1 and K_2 and also in K . Hence, both f_1 and f_2 are continuous and operator norm bounded on K_δ . Further,

$$\begin{aligned} \mu(K \setminus K_\delta) &= \mu\{K \setminus (K_1 \cap K_2)\} = \mu\{(K \setminus K_1) \cup (K \setminus K_2)\} \\ &\leq \mu(K \setminus K_1) + \mu(K \setminus K_2) < \delta/2 + \delta/2 = \delta. \end{aligned}$$

Now, although multiplication is not continuous in the strong operator topology, it is continuous if the operators are restricted to an operator norm bounded subset of $\mathcal{L}(H)$ [2, p. 512]. Since the composition of continuous functions is continuous, it now follows that the function $x \mapsto f_1(x) \cdot f_2(x)$ from K_δ to $\mathcal{L}(H)$ is continuous where $\mathcal{L}(H)$ is equipped with the strong operator topology. This concludes the proof. $\square$

Corollary 2. *Let the hypotheses of Theorem 5 be satisfied except that we assume $f_i: X \rightarrow \mathcal{L}(H)$ is weak operator measurable for $i = 1, \dots, n$. Then the product function $f_1 \cdots f_n$ is also weak operator measurable.*

Proof. Because of the equivalence of strong and weak measurability in Theorem 1, we see that strong operator and weak operator measurability are equivalent in our setting. The result now follows immediately from Theorem 5. $\square$

Remark 4. There are many situations in functional analysis where the proof of a “strong” result that seems not at all clear can be reduced to the relatively easy proof of a “weak” result. Theorem 5 and Corollary 2 reverse this situation. The proof of Theorem 5 hinges on the fact that operator multiplication is continuous in the strong operator topology if the factors are restricted to operator norm bounded subsets of $\mathcal{L}(H)$. The corresponding assertion for the weak operator topology is false [2, p. 512]. So here the “weak” result is reduced to the easier “strong” result.

REFERENCES

1. D. L. Cohn, *Measure theory*, Birkhäuser, Boston, MA, 1980.
2. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Wiley Interscience, New York, 1958.
3. R. P. Feynman, *An operator calculus having applications in quantum electrodynamics*, Phys. Rev. (3) **84** (1951), 108–128.
4. G. W. Johnson, *A bounded convergence theorem for the Feynman integral*, J. Math. Phys. **25** (1984), 1323–1326.
5. G. W. Johnson and M. L. Lapidus, *Generalized Dyson series, generalized Feynman diagrams, the Feynman integral and Feynman's operational calculus*, Mem. Amer. Math. Soc., vol. 62, Amer. Math. Soc., Providence, RI, 1986, pp. 1–78.
6. G. W. Johnson and D. L. Skoug, *A Banach algebra of Feynman integrable functionals with application to an integral equation formally equivalent to Schroedinger's equation*, J. Funct. Anal. **12** (1973), 129–152.
7. ———, *The Cameron-Storvick function space integral: an $\mathcal{L}(L_p, L_{p'})$ theory*, Nagoya Math. J. **60** (1976), 93–137.
8. ———, *Stability theorems for the Feynman integral*, Rend. Circ. Mat. Palermo (2) Suppl. **8** (1985), 361–377.
9. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, revised ed., Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, RI, 1957.
10. M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol. I, *Functional analysis*, revised enlarged ed., Academic Press, New York, 1980.
11. L. Schwartz, *Radon measures on arbitrary topological spaces and cylindrical measures*, Oxford Univ. Press, London, 1973.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEBRASKA-LINCOLN, LINCOLN, NEBRASKA 68588-0323