RATIONALITY OF G/P FOR A NONREDUCED PARABOLIC SUBGROUP-SCHEME P

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ABSTRACT. Let G be a semisimple linear algebraic group over an algebraically closed field K of characteristic $p > 3$. We have described and classified all parabolic subgroup-schemes of G (Trans. Amer. Math. Soc. (to appear)). Here we will show that G/P is a rational projective variety also for a nonreduced parabolic subgroup-scheme P of G .

I. QUOTIENTS

Let K be an algebraically closed field, and let G be a semisimple linear algebraic group over K . Let P be a parabolic subgroup-scheme of G . Then G/P is a scheme (see [J, p. 82 (8)]). Let $\rho: G \rightarrow G/(P_{\text{red}})$ and $\pi: G \rightarrow G/P$ define the quotients. We have a natural map $\iota: G/P_{\text{red}} \rightarrow G/P$ making the following diagram commute:

$$\begin{array}{ccc} (G, O_G) & \xrightarrow{(\rho, \rho^0)} & (G/P_{\text{red}}, O_{G/P_{\text{red}}}) \\ (\pi, \pi^0) \searrow & & \swarrow (\iota, \iota^0) \\ & (G/P, O_{G/P}) \end{array}$$

Notice that $O_{G/P} \cong (\iota_* O_{G/P_{\text{red}}})^P \cong (\pi_* O_G)^P$, where O^P is the subsheaf of P -invariant functions of the sheaf O . Furthermore $G/P(K) \cong G/P_{\text{red}}(K)$, i.e., the underlying spaces of G/P and of G/P_{red} can be identified (see [J, p. 80, I.5.5]).

II. RATIONALITY OF G/P

Definition. A variety is called *rational* if it contains a *dense* open subset isomorphic to affine n -space.

In this section let K be an algebraically closed field of characteristic $p > 0$, and let P be a parabolic subgroup-scheme of G . Let Φ denote the set of roots, let Δ denote the set of simple roots, and let $\tilde{\Delta}$ denote the set of maps

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from Δ to $\mathbb{N}_0 \cup \{\infty\}$. Furthermore, for each $n \in \mathbb{N}_0$, let α_{p^n} denote the local group-scheme $\text{Spec}(K[T]/(T^{p^n}))$, and for each $\beta \in \Phi$, let x_β denote a morphism of algebraic groups from the one-dimensional additive group G_a to G corresponding to β . By abuse of notation we sometimes write α_n instead of α_{p^n} . Then by Theorem III.4 of [W], there exists a unique $\varphi \in \tilde{\Delta}$ such that $P = P_\varphi$, where $P_\varphi = U_\varphi \cdot P_{\text{red}}$, P_{red} equals the (reduced) parabolic subgroup P_I corresponding to $I = \{\beta \in \Delta \mid \varphi(\beta) = \infty\}$, and where

$$U_\varphi = \prod_{\beta \in \Phi^+ - \Phi_I} x_{-\beta}(\alpha_{\varphi(\beta)}).$$

Here Φ_I stands for the roots in Φ generated by I and φ is extended to all of Φ^+ in the following way:

For each $\beta \in \Phi^+$ we can write $\beta = \sum_{\gamma \in \Delta} n_\gamma \cdot \gamma$, with $n_\gamma \in \mathbb{Z}$. Let $E(\beta)$ denote the support of β , that is, the set of all $\gamma \in \Delta$ with $n_\gamma \neq 0$. Then set

$$\varphi(\beta) = \min\{\varphi(\gamma) \mid \gamma \in E(\beta)\}.$$

Let $U_p^- = R_u(P_{\text{red}})^-$, the opposite of the unipotent radical of P_{red} . Then $U_p^- \cdot P_{\text{red}}$ is an open subvariety of G and it contains P . Now $\rho: G \rightarrow G/P_{\text{red}}$ is an open morphism and so is $\pi: G \rightarrow G/P$ since the underlying space of G/P is the same as the underlying space of G/P_{red} . Hence $\pi(U_p^- \cdot P_{\text{red}})$ is an open subscheme of G/P .

We have $\pi(U_p^- \cdot P_{\text{red}}) = (U_p^- \cdot P_{\text{red}})/P = (U_p^- \cdot P_{\text{red}})/(U_\varphi \cdot P_{\text{red}}) \cong U_p^-/U_\varphi$. We will show that U_p^-/U_φ is isomorphic to some affine n -space. The underlying space of U_p^-/U_φ is $(U_p^-/U_\varphi)(K) = U_p^-(K)/U_\varphi(K) = U_p^-(K)$ [J, p. 80], which is homeomorphic to some affine n -space. So the only problem is the structure sheaf of U_p^-/U_φ , i.e., the U_φ -invariant functions of $\mathcal{O}_{U_p^-}$.

The local subgroup-scheme U_φ does not need to be normal in U_p^- .

II.1. Proposition. *Let $G = \text{SL}_3$, and let U^- be the set of lower triangular, unipotent matrices. There exists a local subgroup scheme U_ψ of U^- , which is not normal in U^- .*

Proof. Let $\{\alpha, \beta\} = \Delta$, and $\gamma = \alpha + \beta$. Then $\Phi^+ = \{\alpha, \beta, \gamma\}$. Let $\psi \in \tilde{\Delta}$ be defined by $\psi(\alpha) = 2$, $\psi(\beta) = 1$. Then $E(\gamma) = \{\alpha, \beta\}$. We have to extend ψ to all of Φ^+ by $\psi(\gamma) = \min\{\psi(\delta) \mid \delta \in E(\gamma)\} = \min\{\psi(\alpha), \psi(\beta)\} = 1$. Thus by Chapter III of [W], $U_\psi = x_\alpha(\alpha_{p^2}) \cdot x_\beta(\alpha_{p^1}) \cdot x_\gamma(\alpha_{p^1})$ is a local subgroup-scheme of U^- .

Let $C = K[x, y]/(K[x, y] \cdot x^{p^2} + K[x, y] \cdot y^p)$. Then $\bar{x}^{p^2} = \bar{y}^p = 0$ in C . Let

$$M = x_\beta(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in U^-(C),$$

$$N = x_\alpha(\bar{x}) = \begin{pmatrix} 1 & 0 & 0 \\ \bar{x} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U_\psi(C).$$

Then

$$M \cdot N \cdot M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \bar{x} & 1 & 0 \\ \bar{x} & 0 & 1 \end{pmatrix} \notin U_\psi(C),$$

because $(\bar{x})^{p^{\theta(\gamma)}} = (\bar{x})^{p^1} \neq 0$. Thus $U_\psi(C)$ is not normal in $U^-(C)$, and so U_ψ is not normal in U^- . $\square$

Let U_φ and U be as before Proposition II.1. Then U/U_φ is isomorphic to $\text{Spec}(K[U]^{U_\varphi})$ (see [J, I.5.5(6)]). Now we will compute $K[U]^{U_\varphi}$.

II.2. *Note.* Depending on φ we will introduce an ordering on $\Phi^- - \Phi_I$: Let $\{\beta_1, \dots, \beta_s\} = \Delta^-$ such that $\varphi(\beta_1) \leq \varphi(\beta_2) \leq \dots \leq \varphi(\beta_s)$.

If $\gamma, \delta \in \Phi^- - \Phi_I$ then there are unique $c_i, d_i \in \mathbb{N}_0$ such that

$$\gamma = \sum_{i=1}^s c_i \cdot \beta_i, \quad \delta = \sum_{i=1}^s d_i \cdot \beta_i.$$

We can identify γ with the s -tuple $c = (c_1, c_2, \dots, c_s)$. We can order these s -tuples in a lexicographical order in the following way: Let $m(\gamma) = m(c) = \min\{i \in \langle 1, s \rangle \mid c_i \neq 0\}$. Then $\varphi(\gamma) = \varphi(\beta_{m(\gamma)})$. Define $\gamma < \delta$ if $m(c) < m(d)$, and $\gamma < \delta$ if $m(c) = m(d) = i$ for some $i \in \langle 1, s \rangle$ and if $c_i \geq d_i$, $c_{i+1} \geq d_{i+1}, \dots, c_{i+k-1} \geq d_{i+k-1}$, $c_{i+k} > d_{i+k}$, for some $k \geq 0$.

With this ordering on $\Phi^- - \Phi_I$, let $\Phi^- - \Phi_I = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ with $\gamma_1 < \gamma_2 < \dots < \gamma_k$, for a suitable k . Notice that with this ordering

$$U_\varphi = x_{\gamma_1}(\alpha_{n_1}) \cdot x_{\gamma_2}(\alpha_{n_2}) \cdot \dots \cdot x_{\gamma_k}(\alpha_{n_k}), \quad \text{with } n_1 \leq n_2 \leq \dots \leq n_k, n_i = \varphi(\gamma_i).$$

For an example consider SL_4 , $\Delta^- = \{\alpha, \beta, \gamma\}$, $\varphi(\alpha) = 2$, $\varphi(\beta) = 3$, $\varphi(\gamma) = 1$. Then $I = \{\}$ and $\Phi^- = \{\alpha, \beta, \gamma, \alpha + \beta, \beta + \gamma, \alpha + \beta + \gamma\}$. The corresponding order on Δ^- is $\gamma < \alpha < \beta$, and the corresponding order on Φ^- is

$$\begin{array}{cccccccc} \gamma + \alpha + \beta & < & \gamma + \beta & < & \gamma & < & \alpha + \beta & < & \alpha & < & \beta \\ (1, 1, 1) & < & (1, 0, 1) & < & (1, 0, 0) & < & (0, 1, 1) & < & (0, 1, 0) & < & (0, 0, 1) \\ \varphi(\gamma_i): & \gamma_1 & < & \gamma_2 & < & \gamma_3 & < & \gamma_4 & < & \gamma_5 & < & \gamma_6 \\ & 1 & \leq & 1 & \leq & 1 & \leq & 2 & \leq & 2 & \leq & 3 \end{array}$$

Let us denote $x_{\gamma_i}: G_a \rightarrow U_{\varphi}^-$ also by x_i .

For further reference we give the following formula for any two roots α, β , with $\alpha + \beta \neq 0$, and for any $a, b \in A$, A any K -algebra (for a proof see [SL] or [Sp, 10.1.4]):

$$(*) \quad (x_\alpha(a), x_\beta(b)) = \prod_{\substack{i, j > 1 \\ i\alpha + j\beta \in \Phi}} x_{i\alpha + j\beta}(c_{ij} \cdot a^i \cdot b^j).$$

II.3. **Lemma.** Let $\varphi \in \tilde{\Delta}$. Let $\{\gamma_1, \gamma_2, \dots, \gamma_k\} = \Phi^- - \Phi_I$ be ordered according to φ as described above. Then $U_t = x_1(G_a) \cdot x_2(G_a) \cdot \dots \cdot x_t(G_a)$ is a group for each $t \in \langle 1, k \rangle$.

Proof. Obviously $e = x_1(0) \cdot x_2(0) \cdot \dots \cdot x_t(0) \in U_t$. Let $\gamma, \delta \in \{\gamma_1, \gamma_2, \dots, \gamma_t\}$. From (*) we see that $x_\gamma(u_\gamma)$ and $x_\delta(u_\delta)$ commute up to elements of the form $x_{i\gamma + j\delta}(c_{ij} \cdot u_\gamma^i \cdot u_\delta^j)$, c_{ij} integers for $i, j \geq 1$. We have $i\gamma + j\delta < \delta, \gamma$. By the choice of our ordering $\gamma_1 < \gamma_2 < \dots < \gamma_k$, this implies that $i\gamma + j\delta \in \{\gamma_1, \gamma_2, \dots, \gamma_t\}$, whenever $\delta, \gamma \in \{\gamma_1, \gamma_2, \dots, \gamma_t\}$, and whenever $i\gamma + j\delta \in \Phi^-$. So if we use (*) to compute $x_1(u_1) \cdot x_2(u_2) \cdot \dots \cdot x_t(u_t) \cdot x_1(u'_1) \cdot x_2(u'_2) \cdot \dots \cdot x_t(u'_t)$ for any $u_i, u'_i \in C$, C any K -algebra, then we end up in $U_t(C)$. Thus U_t is multiplicatively closed. By the same argument, the inverse

$x_t(-u_t) \cdot x_{t-1}(-u_{t-1}) \cdots x_1(-u_1) = x_1(v_1) \cdot x_2(v_2) \cdots x_t(v_t) \in U_t(C)$,
for some $v_1, v_2, \dots, v_t \in C$. $\square$

II.4. Lemma. Let $\varphi \in \tilde{\Delta}$. Let $\{\gamma_1, \gamma_2, \dots, \gamma_k\} = \Phi^- - \Phi_I$ be ordered according to φ as described above. Let $n_i = \varphi(\gamma_i)$, and denote the local group-scheme α_{p^n} by α_n . Then $U_{\varphi,t} = x_1(\alpha_{n_1}) \cdot x_2(\alpha_{n_2}) \cdots x_t(\alpha_{n_t})$ is a subgroup-scheme of U_t , for each $t \in \langle 1, k \rangle$.

Proof. $U_{\varphi,t} = U_{\varphi} \cap U_t$ is the intersection of two subgroup-schemes, hence a subgroup-scheme. $\square$

II.5. Proposition. Let $\varphi \in \tilde{\Delta}$. Let $\{\gamma_1, \gamma_2, \dots, \gamma_k\} = \Phi^- - \Phi_I$ be ordered according to φ as described above. Let $t \in \langle 1, k \rangle$. Let C be a K -algebra, and let $x(u) = x_1(u_1) \cdot x_2(u_2) \cdots x_t(u_t) \in U_t(C)$, $x(v) = x_1(v_1) \cdot x_2(v_2) \cdots x_t(v_t) \in U_{\varphi,t}(C)$. Then $x(u) \cdot x(v) = x_1(u_1 + v'_1) \cdot x_2(u_2 + v'_2) \cdots x_t(u_t + v'_t)$ for some $v'_i \in \alpha_{n_i}(C)$, i.e., $(v'_i)^{p^{n_i}} = 0$.

Proof. We will proceed by induction on t . If $t = 1$ then $x_1(u_1) \cdot x_1(v_1) = x_1(u_1 + v_1)$. Now let $t > 1$. Let $x(u) = x_1(u_1) \cdot x_2(u_2) \cdots x_t(u_t) \in U_t(C)$. It suffices to show that for each $r \in \langle 1, t \rangle$ and for $x_r(v_r) \in U_{\varphi,r}(C)$, there exist $v'_i \in C$ such that $x(u) \cdot x_r(v_r) = x_1(u_1 + v'_1) \cdot x_2(u_2 + v'_2) \cdots x_t(u_t + v'_t)$ and $(v'_i)^{p^{n_i}} = 0$.

Case 1. $r = t$. Then

$$\begin{aligned} x(u) \cdot x_r(v_r) &= x_1(u_1) \cdot x_2(u_2) \cdots x_t(u_t) \cdot x_t(v_t) \\ &= x_1(u_1) \cdot x_2(u_2) \cdots x_{t-1}(u_{t-1}) \cdot x_t(u_t + v_t). \end{aligned}$$

Case 2. $r < t$. Then $\varphi(\gamma_r) \leq \varphi(\gamma_t)$ and

$$\begin{aligned} x(u) \cdot x_r(v_r) &= x_1(u_1) \cdot x_2(u_2) \cdots x_t(u_t) \cdot x_r(v_r) \\ &= x_1(u_1) \cdot x_2(u_2) \cdots x_{t-1}(u_{t-1}) \cdot \prod x_{i\gamma_t + j\gamma_r}(c_{ij} \cdot u_t^i \cdot v_r^j) \cdot x_r(v_r) \cdot x_t(u_t), \end{aligned}$$

where the c_{ij} are integers, and where the product is taken over all $i\gamma_t + j\gamma_r$, $i, j > 0$, which are roots.

Suppose $i\gamma_t + j\gamma_r \in \Phi^-$ for some $i, j > 0$. Then $E(i\gamma_t + j\gamma_r) = E(\gamma_t) \cup E(\gamma_r)$ and

$$\begin{aligned} \varphi(i\gamma_t + j\gamma_r) &= \min\{\varphi(\gamma) \mid \gamma \in E(i\gamma_t + j\gamma_r)\} \\ &= \min\{\varphi(\gamma) \mid \gamma \in E(\gamma_t) \cup E(\gamma_r)\} \\ &= \min(\{\varphi(\gamma) \mid \gamma \in E(\gamma_t)\} \cup \{\varphi(\gamma) \mid \gamma \in E(\gamma_r)\}) \\ &= \min\{\varphi(\gamma_t), \varphi(\gamma_r)\} \\ &= \varphi(\gamma_r), \quad \text{since } \varphi(\gamma_r) \leq \varphi(\gamma_t). \end{aligned}$$

Therefore $(c_{ij} \cdot u_t^i \cdot v_r^j)^{p^{\varphi(i\gamma_t + j\gamma_r)}} = (c_{ij} \cdot u_t^i \cdot v_r^j)^{p^{\varphi(\gamma_r)}} = 0$, since $(v_r)^{p^{\varphi(\gamma_r)}} = 0$. This means that $x_{i\gamma_t + j\gamma_r}(c_{ij} \cdot u_t^i \cdot v_r^j) \in U_{\varphi}$. And since $i\gamma_t + j\gamma_r < \gamma_t$, we also have $x_{i\gamma_t + j\gamma_r}(c_{ij} \cdot u_t^i \cdot v_r^j) \in U_{\varphi,t-1}$. Because $r < t$, also $x_r(v_r) \in U_{\varphi,t-1}$, and so is their product $\prod x_{i\gamma_t + j\gamma_r}(c_{ij} \cdot u_t^i \cdot v_r^j) \cdot x_r(v_r)$, since $U_{\varphi,t-1}$ is a group by the above lemma. Now we can apply the induction hypothesis for $t - 1$.

$$\begin{aligned} x(u) \cdot x_r(v_r) &= \left(x_1(u_1) \cdot x_2(u_2) \cdots x_{t-1}(u_{t-1}) \cdot \prod x_{i\gamma_t + j\gamma_r}(c_{ij} \cdot u_t^i \cdot v_r^j) \cdot x_r(v_r) \right) \cdot x_t(u_t) \\ &= (x_1(u_1 + v'_1) \cdot x_2(u_2 + v'_2) \cdots x_{t-1}(u_{t-1} + v'_{t-1})) \cdot x_t(u_t) \end{aligned}$$

with $(v'_i)^{p^{n_i}} = 0$, for $i \in \langle 1, t-1 \rangle$. This proves the proposition. $\square$

II.6. Remark. Proposition II.5 is *not true* for any ordering of the roots. This is the very reason for my ordering of the γ_i 's depending on φ .

Let $K[U_P^-] = K[X_1, X_2, \dots, X_k]$. Then

$$K[U_\varphi] \cong \bigotimes_{i=1}^k K[\alpha_{n_i}] \cong \bigotimes_{i=1}^k K[X_i]/(K[X_i] \cdot X_i^{p^{n_i}}).$$

The underlying space of U_P^-/U_φ equals the underlying space of $U_P^-/((U_\varphi)_{\text{red}}) = U_P^-/(e) = U_P^-$. The sheaf structure on U_P^-/U_φ is the sheaf of U_φ -invariant functions of $O_{U_P^-} : K[U_P^-/U_\varphi] = K[U_P^-]^{U_\varphi}$ (see [J, p. 80 (6)]).

II.7. Theorem. Let $\varphi \in \tilde{\Delta}$. Then $K[U_P^-]^{U_\varphi} = K[X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}}]$.

Proof. Let C be any K -algebra. Let $x(u) = x_1(u_1) \cdot x_2(u_2) \cdots x_k(u_k) \in U_P^-(C)$, and let $x(v) = x_1(v_1) \cdot x_2(v_2) \cdots x_k(v_k) \in U_\varphi(C)$. By Proposition II.5 we have $x(u) \cdot x(v) = x_1(u_1 + v'_1) \cdot x_2(u_2 + v'_2) \cdots x_k(u_k + v'_k)$ for some $v'_i \in \alpha_{n_i}(C)$, i.e., $(v'_i)^{p^{n_i}} = 0$. Let $i \in \langle 1, k \rangle$. Then

$$\begin{aligned} X_i^{p^{n_i}}(x(u) \cdot x(v)) &= X_i^{p^{n_i}}(x_i(u_i + v'_i)) = (u_i + v'_i)^{p^{n_i}} \\ &= (u_i)^{p^{n_i}} + (v'_i)^{p^{n_i}} = (u_i)^{p^{n_i}} = X_i^{p^{n_i}}(x(u)). \end{aligned}$$

Hence $X_i^{p^{n_i}} \in K[U_P^-]^{U_\varphi}$ for all $i \in \langle 1, k \rangle$; $K[X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}}] \subset K[U_P^-]^{U_\varphi}$.

To prove equality, we will compare certain dimensions of vector spaces. Let $K[U_P^-]^{U_\varphi} = K[X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}}, g_1, g_2, \dots, g_n, \dots] = B$, with g_i in the distinguished maximal ideal of $K[U_P^-]$. Now U_φ is the fiber over the distinguished point of U_P^-/U^φ :

$$\begin{array}{ccc} U_\varphi = \text{Spec}(K) \times_{U_P^-/U_\varphi} U_P^- & \rightarrow & U_P^- \\ \downarrow & & \downarrow \\ \text{Spec}(K) & \rightarrow & U_P^-/U_\varphi \end{array}$$

So we have

$$\begin{aligned} K[U_\varphi] &\cong K \otimes_B K[U_P^-] \cong K \otimes_B K[X_1, X_2, \dots, X_k] \\ &\cong K[X_1, X_2, \dots, X_k]/(X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}}, g_1, g_2, \dots). \end{aligned}$$

If $g_i \notin (X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}})$ then the K -linear dimension of

$$K[X_1, X_2, \dots, X_k]/(X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}}, g_1, g_2, \dots)$$

is strictly less than the K -linear dimension of

$$K[X_1, X_2, \dots, X_k]/(X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}}),$$

which equals $p^{n_1} \cdot p^{n_2} \cdots p^{n_k} = \dim(K[U_\varphi])$. This would be a contradiction.

So $K[X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}}] = K[U_P^-]^{U_\varphi}$. $\square$

II.8. Theorem. Let G be a semisimple linear algebraic group over an algebraically closed field of characteristic $p > 3$. Let P be a parabolic subgroup-scheme of G . Then G/P is a rational projective variety.

Proof. We know that G/P is a scheme and that

$$\pi(U_P^- \cdot P_{\text{red}}) = (U_P^- \cdot P_{\text{red}})/P = (U_P^- \cdot P_{\text{red}})/(U_\varphi \cdot P_{\text{red}}) \cong U_P^-/U_\varphi$$

is an open subset of G/P homeomorphic to affine K -space $\mathbb{A}^k(K)$.

Now

$$\begin{aligned} O_{G/P}(\pi(U_P^- \cdot P_{\text{red}})) &= (\pi_*(O_G^P))(\pi(U_P^- \cdot P_{\text{red}})) = O_G^P(\pi^{-1}(\pi(U_P^- \cdot P_{\text{red}}))) \\ &= O_G^P(U_P^- \cdot P_{\text{red}}) = O_G(U_P^- \cdot P_{\text{red}})^P \\ &\cong (O_{U_P^-}(U_P^-))^{U_\varphi} = K[U_P^-]^{U_\varphi} \\ &= K[X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}}], \end{aligned}$$

by Theorem II.7.

Thus

$$(\pi(U_P^- \cdot P_{\text{red}}), O_{G/P}|_{\pi(U_P^- \cdot P_{\text{red}})}) \cong \text{Spec } K[X_1^{p^{n_1}}, X_2^{p^{n_2}}, \dots, X_k^{p^{n_k}}] \cong (\mathbb{A}^k, O_{\mathbb{A}^k}),$$

and G/P is a rational projective variety. $\square$

REFERENCES

- [SL] R. Steinberg, *Lectures on Chevalley groups*, Yale Univ. Press, New Haven, CT, 1968.
- [Sp] T. A. Springer, *Linear algebraic groups*, Birkhäuser, Boston, Basel, and Stuttgart, 1981.
- [W] C. Wenzel, *Classification of all parabolic subgroup schemes of a semi-simple linear algebraic group over an algebraically closed field*, Trans. Amer. Math. Soc. (to appear).
- [J] J. C. Jantzen, *Representations of algebraic groups*, Academic Press, New York, 1987.

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