A RANDOM FIXED POINT THEOREM FOR MULTIVALUED NONEXPANSIVE OPERATORS IN UNIFORMLY CONVEX BANACH SPACES

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ABSTRACT. Let (Ω, Σ) be a measurable space with Σ a sigma-algebra of subsets of Ω , and let C be a nonempty, bounded, closed, convex, and separable subset of a uniformly convex Banach space X . It is shown that every multivalued nonexpansive random operator $T: \Omega \times C \rightarrow K(C)$ has a random fixed point, where $K(C)$ is the family of all nonempty compact subsets of C endowed with the Hausdorff metric induced by the norm of X .

The purpose of this note is to prove the random version of the following celebrated deterministic result due to Lim [12].

Theorem L [12]. *Let X be a uniformly convex Banach space, let C be a nonempty, bounded, closed, convex subset of X , and let $K(C)$ be the collection of all nonempty compact subsets of C endowed with the Hausdorff metric. Then every multivalued nonexpansive mapping $T: C \rightarrow K(C)$ has a fixed point.*

We begin with establishing some preliminaries. By (Ω, Σ) we denote a measurable space with Σ a sigma-algebra of subsets of Ω . Let (X, d) be a metric space. Then we denote by $K(X)$ the family of all nonempty compact subsets of X and by H the Hausdorff metric on $K(X)$ induced by the metric d of X ; that is, for A, B in $K(X)$,

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

where $d(x, E)$ is the distance from a point $x \in X$ to a subset $E \subset X$, i.e., $d(x, E) = \inf\{d(x, y) : y \in E\}$. Recall that a multivalued mapping $T: X \rightarrow K(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq d(x, y)$$

for all x, y in X and that a point x in X is said to be a fixed point of T if $x \in Tx$. A multivalued operator $f: \Omega \rightarrow X$ is said to be (Σ) -measurable if, for any open subset B of X , $f^{-1}(B) = \{\omega \in \Omega : f(\omega) \cap B \neq \emptyset\} \in \Sigma$. Let M be

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a nonempty closed subset of X . Then a multivalued operator $f: \Omega \times M \rightarrow X$ is called a random operator if, for each fixed $x \in M$, the map $f(\cdot, x): \Omega \rightarrow X$ is measurable. A single-valued operator $x: \Omega \rightarrow X$ is said to be a random fixed point of a multivalued random operator $f: \Omega \times M \rightarrow X$ if x is measurable and $x(\omega) \in M \cap f(\omega, x(\omega))$ for all $\omega \in \Omega$. A multivalued random operator $f: \Omega \times M \rightarrow X$ is said to be continuous (resp. nonexpansive, etc.) if for each fixed $\omega \in \Omega$, the multivalued map $f(\omega, \cdot): M \rightarrow X$ is continuous (resp. nonexpansive, etc.). We shall use the notion of a “universal net.”

A net $\{x_\alpha\}$ in a set S is said to be universal if for each subset U of S , either $\{x_\alpha\}$ is eventually in U or $\{x_\alpha\}$ is eventually in $S \setminus U$. The following facts are pertinent [8, p. 81]:

- (a) Every net in a set has a universal subset.
- (b) If $f: S_1 \rightarrow S_2$ and $\{x_\alpha\}$ is a universal net in S_1 , then $\{f(x_\alpha)\}$ is a universal net in S_2 .
- (c) If S is compact and $\{x_\alpha\}$ is a universal net in S , then $\lim_\alpha x_\alpha$ exists.

The reader is referred to Kirk [9], Kirk and Massa [10], Kirk and Yanez [11], and Aksoy and Khamsi [1] for applications of universal nets to fixed point theory.

We now state and prove the main result of this note, which can be viewed as the stochastic version of Lim’s theorem mentioned above. See [2, 6, 7, 15, 17] and references therein for more details of random fixed point theory. We note that previous existence results for random fixed points of multivalued nonexpansive operators are obtained only in a space that satisfies Opial’s condition [16]. A Banach space X is said to satisfy Opial’s condition if for any sequence $\{x_n\}$ in X , the condition $x_n \rightarrow x \in X$ weakly implies that $\liminf_{n \rightarrow \infty} \|x_n - y\| > \liminf_{n \rightarrow \infty} \|x_n - x\|$ for all $y \in X$, $y \neq x$. It is known [16] that all Hilbert spaces and the l^p spaces ($1 < p < \infty$) enjoy this property; however, the L^p spaces do not unless $p = 2$.

Theorem. *Let (Ω, Σ) be a measurable space with Σ a sigma-algebra of subsets of Ω , let C be a nonempty, bounded, closed, convex, and separable subset of a uniformly convex Banach space X , and let $T: \Omega \times C \rightarrow K(C)$ be a multivalued nonexpansive random operator. Then T has a random fixed point.*

Proof. Let $v \in C$ be fixed and set, for each integer $n \geq 1$,

$$T_n(\omega, x) = \left(1 - \frac{1}{n}\right) T(\omega, x) + \frac{1}{n}v, \quad \omega \in \Omega, x \in C.$$

Then each T_n is a multivalued random contraction and hence has a random fixed point $x_n(\omega)$ by Itoh [6]. Let $\{n_\alpha\}$ be a universal subnet of the positive integers $\{n\}$. Define a function $f: \Omega \times C \rightarrow R^+ := [0, \infty)$ by

$$f(\omega, x) = \lim_\alpha \|x_{n_\alpha}(\omega) - x\|.$$

Since $\{x_{n_\alpha}(\omega)\}$ is countable, it is easily seen that $f(\omega, x)$ is measurable. On the other hand, since the space X is uniformly convex and C is weakly compact convex, there exists a unique point $x(\omega) \in C$ such that

$$f(\omega, x(\omega)) = \inf_{x \in C} f(\omega, x) =: r(\omega).$$

This point $x(\omega)$ is referred to as, according to terminology of Edelstein [4] and Lim [14], the asymptotic center of the net $\{x_{n_\alpha}(\omega)\}$ in C . Lim [13], Goebel

[5], and Kirk and Massa [10] actually proved that for each fixed $\omega \in \Omega$, $x(\omega)$ is a fixed point of the map $T(\omega, \cdot)$. So to complete the proof of the theorem, it remains only to show the measurability of $x(\cdot)$. To this end, let $\{u_n\}_{n=1}^\infty$ be a countable dense subset of C . (This can be done since C is separable.) Then one easily sees that $r(\omega) = \inf_{n \geq 1} f(\omega, u_n)$. We first show that $r: \Omega \rightarrow R^+$ is measurable. In fact, for every real number s , we have

$$\begin{aligned} \{\omega \in \Omega: r(\omega) \geq s\} &= \left\{ \omega \in \Omega: \inf_{n \geq 1} f(\omega, u_n) \geq s \right\} \\ &= \bigcap_{n=1}^{\infty} \{\omega \in \Omega: f(\omega, u_n) \geq s\} \in \Sigma \end{aligned}$$

since f is measurable. This proves the measurability of $r(\omega)$. Now for each integer $k \geq 1$, we let

$$A_k(\omega) = \left\{ x \in C: f(\omega, x) \leq r(\omega) + \frac{1}{k} \right\}.$$

It then follows that $A_k: \Omega \rightarrow C$ is measurable and for each $\omega \in \Omega$, $A_k(\omega)$ is a weakly compact convex subset of C . It is readily seen that

$$(1) \quad \bigcap_{k=1}^{\infty} A_k(\omega) = \{x(\omega)\}.$$

Since C is separable, the weak topology on C is metrizable. Let d_w be the metric produced by the weak topology on C and let H_w be the Hausdorff metric induced by d_w . We show that

$$(2) \quad \lim_{k \rightarrow \infty} H_w(A_k(\omega), x(\omega)) = 0.$$

Indeed, since $\bigcap_{k=1}^{\infty} A_k(\omega) = x(\omega)$ by (1), the limit in (2) exists and we denote it by $h(\omega)$. If $h(\omega) > 0$, then observing

$$H_w(A_k(\omega), x(\omega)) = \sup\{d_w(y, x(\omega)): y \in A_k(\omega)\},$$

we have for each k a $y_k \in A_k(\omega)$ such that

$$(3) \quad d_w(y_k, x(\omega)) > \frac{1}{2}h(\omega).$$

Since C is weakly compact, there is a subsequence $\{y_{k'}\}$ of $\{y_k\}$ converging weakly to some $y \in C$, i.e., $d_w(y_{k'}, y) \rightarrow 0$ as $k' \rightarrow \infty$. Since $\{A_k(\omega)\}$ is decreasing, $y \in \bigcap_{k=1}^{\infty} A_k(\omega) = x(\omega)$. It thus follows that $d_w(y_{k'}, x(\omega)) \rightarrow 0$ as $k' \rightarrow \infty$. This contradicts (3), and (2) is proven. By Itoh [6, Proposition 1], it follows that $x(\omega)$ is weakly measurable, i.e., for every $x^* \in X^*$, the dual space of X , $x^*x(\omega)$ is measurable as a numerically valued function on Ω . Again since C is separable, by Bharucha-Reid [3, Theorem 1.2], $x(\omega)$ is measurable. The proof of the theorem, therefore, is complete.

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