A CHARACTERIZATION OF HARDY-ORLICZ SPACES ON PLANAR DOMAINS

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ABSTRACT. In the paper we prove that for a wide class of bounded domains D in $\mathbb{C}$, a holomorphic function f is in the Hardy-Orlicz space $H_\phi(D)$ if and only if

$$\iint_D \delta(z) \phi''(\log |f(z)|) \frac{|f'(z)|^2}{|f(z)|^2} dx dy < \infty,$$

where $\delta(z)$ denotes the distance from z to the boundary of D and ϕ is a strongly convex function on $(-\infty, \infty)$ for which $\phi''(t)$ exists for all t .

1. INTRODUCTION

For $0 < p < \infty$, the Hardy classes H^p in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ are defined as the set of holomorphic functions f in U for which

$$\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

It is well known that this is equivalent to the subharmonic function $|f(z)|^p$ having a harmonic majorant on U . In [Y1] Yamashita proved that a holomorphic function f on U is in H^p , $0 < p < \infty$, if and only if

$$(1.1) \quad \iint_U (1 - |z|) |f(z)|^{p-2} |f'(z)|^2 dx dy < \infty.$$

The case $p = 2$ had previously been noted in [Z, Remark(a), p. 208]. Analogous problems have also been considered by Kobayashi and Suita in [KS] for H^p spaces on a nonparabolic Riemann surface R . In a subsequent paper [Y2] Yamashita proved that if $f \in H^p$, then

$$(1.2) \quad \lim_{r \rightarrow 1} (1 - r) \iint_{|z| \leq r} |f(z)|^{p-2} |f'(z)|^2 dx dy = 0.$$

The purpose of this note is to obtain analogous results for functions in the Hardy-Orlicz class H_ϕ on planar domains in $\mathbb{C}$ that have a Green's function G . For convenience we will take the Green's function to be positive and hence superharmonic.

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As in [R2], a function ϕ on $(-\infty, \infty)$ is said to be *strongly convex* if ϕ is nonnegative, convex, and nondecreasing with $\phi(t)/t \rightarrow \infty$ as $t \rightarrow \infty$. Let D be any domain in $\mathbb{C}$. A holomorphic function f on D is said to be in the Hardy-Orlicz class $H_\phi(D)$ if there exists a harmonic function h on D such that

$$(1.3) \quad \phi(\log |f(z)|) \leq h(z)$$

for all $z \in D$. Taking $\phi(t) = e^{pt}$ gives the usual Hardy space $H^p(D)$ as considered in [R1]. The main result of the paper is

Theorem. *Let D be a domain in $\mathbb{C}$ that has a Green's function G , and let ϕ be a strongly convex function such that $\phi''(t)$ exists for all $t \in \mathbb{R}$. A holomorphic function $f \in H_\phi(D)$ if and only if*

$$(1.4) \quad \iint_D G(t_o, z) \phi''(\log |f(z)|) \frac{|f'(z)|^2}{|f(z)|^2} dx dy < \infty$$

for some $t_o \in D$. Furthermore, if $f \in H_\phi(D)$ then

$$(1.5) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \iint_{\Omega_\varepsilon} \phi''(\log |f(z)|) \frac{|f'(z)|^2}{|f(z)|^2} dx dy = 0,$$

where $\Omega_\varepsilon = \{z \in D : G(t_o, z) \geq \varepsilon\}$.

As a consequence of the theorem, we will show that for a wide class of bounded domains D , a holomorphic function $f \in H_\phi(D)$ if and only if

$$(1.6) \quad \iint_D \delta(z) \phi''(\log |f(z)|) \frac{|f'(z)|^2}{|f(z)|^2} dx dy < \infty,$$

where $\delta(z)$ denotes the distance from z to the boundary of D . Furthermore, if $f \in H_\phi$ then

$$(1.7) \quad \lim_{s \rightarrow 0} s \iint_{D_s} \phi''(\log |f(z)|) \frac{|f'(z)|^2}{|f(z)|^2} dx dy = 0,$$

where $D_s = \{z \in D : \delta(z) \geq s\}$. For the unit disc U , $\delta(z) = (1 - |z|)$, and thus (1.6) and (1.7) with $\phi(t) = e^{pt}$ imply the results of Yamashita.

If f is holomorphic in D and ϕ is strongly convex, then the function $g(z) = \phi(\log |f(z)|)$ is a nonnegative subharmonic function on D . In §2 we will use potential theory methods to obtain a characterization of those subharmonic functions f on D that have a harmonic majorant in terms of the Riesz measure μ_f of f . The theorem is proved in §3 by computing the Riesz measure of $g(z) = \phi(\log |f(z)|)$. In §4 we also give a characterization of the Nevanlinna class $N(D)$ and indicate how the results can be extended to Hardy-Orlicz spaces H_ϕ , where $\phi(\log |f(z)|) = \phi(\log^+ |f(z)|)$.

2. HARMONIC MAJORANTS OF SUBHARMONIC FUNCTIONS

For a domain D in $\mathbb{C}$, we denote by $S(D)$ the class of subharmonic functions f on D for which f has a harmonic majorant on D . It is well known that if a subharmonic function f has a harmonic majorant on D , then it has a least harmonic majorant on D . For $f \in S(D)$, we denote the least harmonic

majorant of f by h_f . In the definition of subharmonic we exclude the case $f \equiv -\infty$.

Let f be subharmonic on D . The *Riesz measure* of f is the nonnegative regular Borel measure μ_f on D satisfying

$$(2.1) \quad \iint_D \varphi d\mu_f = \frac{1}{2\pi} \iint_D f \nabla^2 \varphi d\lambda$$

for all $\varphi \in C_c^\infty(D)$, where $\nabla^2 \varphi$ denotes the Laplacian of φ and λ is two-dimensional area measure on D . If f is C^2 , then by Green's identity, $d\mu_f = \frac{1}{2\pi} \nabla^2 f d\lambda$.

By a regular exhaustion $\{\Omega_n\}$ of D we mean a sequence of domains Ω_n , with compact boundary Γ_n such that $\Omega_n \cup \Gamma_n \subset \Omega_{n+1}$, $\bigcup_1^\infty \Omega_n = D$, and such that the Dirichlet problem is solvable for Ω_n . By [F, Proposition 1.5.3], each domain in $\mathbb{C}$ has a regular exhaustion. For each n , let G_n denote the Green's function of Ω_n . If f is subharmonic on D with Riesz measure μ_f , then by the Riesz Decomposition Theorem

$$(2.2) \quad f(z) = - \iint_{\Omega_n} G_n(z, w) d\mu_f(w) + h_n(z),$$

for all $z \in \Omega_n$, where $h_n(z)$ is the least harmonic majorant of f on Ω_n . If the boundary Γ_n of Ω_n consists of a finite number of disjoint simple closed analytic curves, then

$$(2.3) \quad h_n(z) = \frac{1}{2\pi} \int_{\Gamma_n} f(t) \frac{\partial G_n}{\partial \eta}(z, t) ds(t),$$

where the normal derivative of G_n is taken along the interior normal η . For proofs of the above the reader is referred to [T].

The following proposition, which undoubtedly has been proved elsewhere, is included for completeness.

Proposition 1. *Let f be subharmonic in D and let μ_f be the Riesz measure of f . Then $f \in S(D)$ if and only if there exists $t_o \in D$ such that*

$$(2.4) \quad \sup_n \iint_{\Omega_n} G_n(t_o, z) d\mu_f(z) < \infty$$

for any regular exhaustion $\{\Omega_n\}$ of D . If the domain D has a Green's function G , then $f \in S(D)$ if and only if there exists $t_o \in D$ such that

$$(2.5) \quad \iint_D G(t_o, z) d\mu_f(z) < \infty.$$

Proof. Suppose $f \in S(D)$. Let h be any harmonic majorant of f on D , and let $\{\Omega_n\}$ be a regular exhaustion of D . For each n , let h_n be the least harmonic majorant of f on Ω_n given by (2.2). Since $f \leq h$ on Ω_n , we have that $h_n(z) \leq h(z)$ for all $z \in \Omega_n$. Since $f(z) > -\infty$ a.e. on D , there exists $t_o \in \Omega_1$ such that $f(t_o) > -\infty$. For this t_o , by (2.2),

$$\iint_{\Omega_n} G_n(t_o, z) d\mu_f(z) = h_n(t_o) - f(t_o) \leq h(t_o) - f(t_o),$$

which proves (2.4). If the domain D itself has a Green's function G , then $G_n(t_o, z) \leq G(t_o, z)$ for all $z \in \Omega_n$ and $G(t_o, z) = \lim_{n \rightarrow \infty} G_n(t_o, z)$. Thus (2.5) follows by the Monotone Convergence Theorem.

Conversely, suppose (2.4) holds for some $t_o \in D$ and any regular exhaustion $\{\Omega_n\}$ of D . Without loss of generality we may assume $t_o \in \Omega_n$ for all n . For each n let h_n be the least harmonic majorant of f on Ω_n as given by (2.2). Clearly, if $n < m$, then $h_n(z) \leq h_m(z)$ for all $z \in \Omega_n$. By hypothesis, there exists a constant M such that $h_n(t_o) \leq M + f(t_o)$ for all n . Hence by Harnack's Theorem, $h(z) = \lim_{n \rightarrow \infty} h_n(z)$ is harmonic on D , and thus $f \in S(D)$.

Let D be a domain with Green's function G . For $t \in D$ and $\varepsilon > 0$, let

$$\Omega_\varepsilon(t) = \{z \in D : G(t, z) \geq \varepsilon\}.$$

Proposition 2. *Let D be a domain with Green's function G , and let $f \in S(D)$ with Riesz measure μ_f . Then*

$$(2.6) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon \mu_f(\Omega_\varepsilon(t)) = 0 \quad \text{for all } t \in D.$$

Proof. Let $t_o \in D$ be arbitrary and set

$$(2.7) \quad g(z) = \iint_D G(z, w) d\mu_f(w).$$

Since $s \rightarrow \mu_f(\Omega_s(t_o))$ is the distribution function of $G(t_o, w)$,

$$\int_0^\infty \mu_f(\Omega_s(t_o)) ds = \iint_D G(t_o, w) d\mu_f(w) = g(t_o).$$

Therefore,

$$\varepsilon \mu_f(\Omega_\varepsilon(t_o)) \leq \int_0^\varepsilon \mu_f(\Omega_s(t_o)) ds \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

whenever $g(t_o) < \infty$. When $g(t_o) = \infty$, choose α sufficiently large so that $\Omega_\alpha(t_o)$ is compact, and let ν_α be the measure defined by $(1 - \chi_\alpha)\mu_f$, where χ_α is the characteristic function of $\Omega_\alpha(t_o)$. Let g_α denote the corresponding function defined by (2.7) for the measure ν_α . Since g_α is harmonic at t_o , it follows that $g_\alpha(t_o) < \infty$, and hence as above, $\varepsilon \nu_\alpha(\Omega_\varepsilon(t_o)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since $\mu_f(\Omega_\alpha(t_o))$ is finite and

$$\varepsilon \mu_f(\Omega_\varepsilon(t_o)) = \varepsilon \mu_f(\Omega_\alpha(t_o)) + \varepsilon \nu_\alpha(\Omega_\varepsilon(t_o)),$$

$\lim_{\varepsilon \rightarrow 0} \varepsilon \mu_f(\Omega_\varepsilon(t_o)) = 0$, which proves the result.

We now assume that D is a bounded domain in $\mathbb{C}$ with Green's function G . For $z \in D$, let $\delta(z)$ denote the distance from z to ∂D , the boundary of D . We will assume that the Green's function G satisfies the following: for each $t_o \in D$, there exist constants c_1 and c_2 , depending only on t_o and D , such that

$$(2.8) \quad \begin{aligned} & \text{(i) } c_1 \delta(z) \leq G(t_o, z) \quad \text{for all } z \in D, \\ & \text{(ii) } G(t_o, z) \leq c_2 \delta(z) \quad \text{for all } z \in D \sim D(t_o, \tfrac{1}{2}\delta(t_o)). \end{aligned}$$

In the case where the boundary of D consists of a finite number of disjoint, simple, closed analytic curves, (2.8) was proved in [W]. If the boundary of D is C^2 , then the inequalities can be established by comparing the Green's function G to the Green's function of discs which are internally tangent to obtain (i) and to the Green's function of the complement of externally tangent discs to obtain (ii).

Proposition 3. *Let D be a bounded domain with Green's function G satisfying (2.8), and let f be subharmonic on D with Riesz measure μ_f . Then $f \in S(D)$ if and only if*

$$(2.9) \quad \iint_D \delta(z) d\mu_f(z) < \infty.$$

Furthermore, if $f \in S(D)$, then

$$(2.10) \quad \lim_{s \rightarrow 0^+} s \mu_f(D_s) = 0,$$

where $D_s = \{z \in D : \delta(z) \geq s\}$.

Proof. Suppose $f \in S(D)$ and $t_o \in D$ is such that (2.5), and hence also (2.6), holds. Then (2.9) follows by (2.8(i)), and since $D_s \subset \Omega_{c_1 s}(t_o)$, we obtain (2.10).

Conversely, suppose (2.9) holds. Let $p_o \in D$ be arbitrary and let $K = \overline{D}(p_o, \frac{1}{2}\delta(p_o))$. Let

$$g_1(z) = \iint_K G(z, w) d\mu_f(w).$$

Since K is compact, g_1 is superharmonic on D . In particular, $g_1(z) < \infty$ a.e. on D . Also, let

$$g_2(z) = \iint_{D \sim K} G(z, w) d\mu_f(w).$$

Since $G(p_o, w) \leq c_2 \delta(w)$, we have $g_2(p_o) < \infty$ by hypothesis. Thus $g_2(z)$ is also superharmonic on D and hence finite a.e. Thus there exists $t_o \in D$ such that both g_1 and g_2 are finite at t_o . Therefore μ_f satisfies (2.5), which proves the result.

3. PROOF OF THE THEOREM

Let D be a domain in $\mathbb{C}$ with Green's function G . Suppose $f \in H_\phi(D)$, where ϕ is a strongly convex function for which $\phi''(t)$ exists for all $t \in \mathbb{R}$. Since ϕ is nonnegative and nondecreasing, $\phi(-\infty) = \lim_{t \rightarrow -\infty} \phi(t)$ exists. Also, since $\phi''(t)$ exists for all $t \in \mathbb{R}$ and $\phi''(t) \geq 0$, it follows that $\phi'(t)$ is continuous, nondecreasing, and nonnegative on $\mathbb{R}$. As a consequence, $\lim_{t \rightarrow -\infty} \phi'(t)$ exists. Furthermore, since $\phi(t) = \phi(-\infty) + \int_{-\infty}^t \phi'(x) dx$, we have

$$(3.1) \quad \lim_{t \rightarrow -\infty} \phi'(t) = 0.$$

Let $g(z) = \phi(\log |f(z)|)$. Then g is a nonnegative subharmonic function on D . To prove the theorem it suffices to show that the Riesz measure μ_g is absolutely continuous and is given by

$$(3.2) \quad d\mu_g(z) = \frac{1}{2\pi} \phi''(\log |f(z)|) \frac{|f'(z)|^2}{|f(z)|^2} d\lambda(z).$$

For convenience, set

$$f_\phi^\#(z) = \phi''(\log |f(z)|) \frac{|f'(z)|^2}{|f(z)|^2}.$$

Let $D_o = \{z \in D : f(z) = 0\}$. By a routine computation, on $D \sim D_o$, $\nabla^2 g(z) = f_\phi^\#(z)$. Thus to show that μ_g is absolutely continuous and is given by the above, it suffices to show that

$$(3.3) \quad \iint_D \phi f_\phi^\# d\lambda = \iint_D g \nabla^2 \phi d\lambda$$

for all $\phi \in C_c^\infty(D)$.

Suppose $\phi \in C_c^\infty(D)$. Since ϕ has compact support and the zeros of f are isolated, we can assume without loss of generality that the support of ϕ contains exactly one zero of f , say p_o , which for convenience we will take to be zero. Let $r > 0$ be such that $\overline{D}(0, r) \subset D$ and that $f(z) \neq 0$ for all $z \in \overline{D}(0, r)$, $z \neq 0$. Let $D_r = D \sim \overline{D}(0, r)$ and let $S_r = \partial D(0, r)$. Then by Green's identity,

$$\iint_{D_r} [\phi \nabla^2 g - g \nabla^2 \phi] d\lambda = \int_{S_r} \left[\phi \frac{\partial g}{\partial \eta} - g \frac{\partial \phi}{\partial \eta} \right] ds.$$

We now show that the term on the right goes to zero as $r \rightarrow 0$. Since $\frac{\partial \phi}{\partial \eta}$ and $g(re^{i\theta})$ are bounded in a neighborhood of 0, we have

$$\lim_{r \rightarrow 0} \int_{S_r} g \frac{\partial \phi}{\partial \eta} ds = 0.$$

Consider

$$\int_{S_r} \phi \frac{\partial g}{\partial \eta} ds = -r \int_0^{2\pi} \phi(re^{i\theta}) \frac{\partial}{\partial r} \phi(\log |f(re^{i\theta})|) d\theta.$$

But $\frac{\partial}{\partial r} \phi(\log |f(re^{i\theta})|) = \phi'(\log |f(re^{i\theta})|) \frac{\partial}{\partial r} \log |f(re^{i\theta})|$. Suppose f has a zero of order $m \geq 1$ at 0. Then $f(w) = w^m h(w)$ where h is holomorphic in a neighborhood of 0 with $h(0) \neq 0$. Therefore $\log |f(re^{i\theta})| = m \log r + \log |h(re^{i\theta})|$, and thus

$$\frac{\partial}{\partial r} \log |f(re^{i\theta})| = \frac{m}{r} + O(1).$$

Therefore

$$\left| \phi(re^{i\theta}) \frac{\partial}{\partial r} \phi(\log |f(re^{i\theta})|) \right| \leq \frac{M}{r} \phi'(\log |f(re^{i\theta})|).$$

By (3.1), $\phi'(\log |f(re^{i\theta})|) \rightarrow 0$ as $r \rightarrow 0$. Thus

$$(3.4) \quad \lim_{r \rightarrow 0} \left| \int_{S_r} \phi \frac{\partial g}{\partial \eta} ds \right| = 0.$$

Finally, since ϕ has compact support and g is bounded on the support of ϕ ,

$$\lim_{r \rightarrow 0} \iint_{D_r} g \nabla^2 \phi d\lambda = \iint_D g \nabla^2 \phi d\lambda.$$

Therefore, $\lim_{r \rightarrow 0} \iint_{D_r} \phi f_\phi^\# d\lambda$ exists and is finite, and as a consequence, (3.3) holds for all $\phi \in C_c^\infty(D)$. Therefore μ_g is absolutely continuous and is given by (3.2).

Remark. If the Green's function G of D satisfies (2.8), then $f \in H_\phi(D)$ if and only if (1.6) holds and if this is the case then (1.7) holds also.

4. RELATED RESULTS

The methods of the preceding section are not restricted to the Hardy-Orlicz spaces H_ϕ where $\phi''(t)$ exists for all t . They can also be applied to other spaces of holomorphic functions, including the Nevanlinna class $N(D)$ and Hardy-Orlicz spaces H_ϕ , where $\phi(\log |f(z)|) = \phi(\log^+ |f(z)|)$ and ϕ'' does not exist at 0. As in [R1], let $\text{Log}^+(D)$ denote the set of holomorphic functions f on D for which $\log^+ |f(z)| = \max\{\log |f(z)|, 0\}$ has a harmonic majorant on D . The space $\text{Log}^+(D)$ is also referred to as the Nevanlinna class $N(D)$. From the inequality

$$(4.1) \quad \log^+ x \leq \log(1+x) \leq \log^+ x + \log 2 \quad (x \geq 0),$$

$f \in \text{Log}^+(D)$ if and only if the subharmonic function $g(z) = \log(1 + |f(z)|)$ is in $S(D)$.

As in §3, let $D_o = \{z \in D : f(z) = 0\}$. Then on $D \sim D_o$,

$$\nabla^2 g(z) = \frac{1}{(1 + |f(z)|)^2} \frac{|f'(z)|^2}{|f(z)|}.$$

If p_o is a zero of f of order m , then in a neighborhood of p_o ,

$$g(p_o + re^{i\theta}) = \log(1 + r^m |h(p_o + re^{i\theta})|),$$

where h is analytic at p_o with $h(p_o) \neq 0$. Hence for r sufficiently small, $|\frac{\partial}{\partial r} g(p_o + re^{i\theta})| = O(r^{m-1})$, and consequently,

$$\left| \int_{S_r} \varphi \frac{\partial g}{\partial \eta} ds \right| = O(r^m).$$

Thus, as in §3,

$$d\mu_g(z) = \frac{1}{2\pi} \frac{1}{(1 + |f(z)|)^2} \frac{|f'(z)|^2}{|f(z)|} d\lambda(z).$$

As a consequence we obtain

Proposition 4. Let D be a domain with Green's function G . Then $f \in \text{Log}^+(D)$ if and only if

$$\iint_D G(t_o, z) \frac{|f'(z)|^2}{(1 + |f(z)|)^2 |f(z)|} d\lambda(z) < \infty$$

for any $t_o \in D$.

The methods can also be extended to Hardy-Orlicz spaces where the strongly convex function ϕ satisfies

- (1) $\phi(t) = 0$ for all $t < 0$ with $\phi(0) = \phi'(0) = 0$,
- (2) $\phi''(t)$ exists for all $t > 0$, and
- (3) $\phi(2t) \leq c\phi(t)$ for some positive constant c and all $t > 0$.

The function ϕ_α defined for $\alpha > 1$ by $\phi_\alpha(t) = t^\alpha$ for $t > 0$ and $\phi_\alpha(t) = 0$ for $t \leq 0$ is an example of such a function. By property (3) and equation (4.1), $f \in H_\phi(D)$ if and only if the subharmonic function $g(z) = \phi(\log(1 + |f(z)|))$

has a harmonic majorant on D . As in §3, the Riesz measure μ_g of g is given by

$$d\mu_g(z) = \frac{1}{2\pi} \left[\phi''(\log(1 + |f(z)|)) + \frac{\phi'(\log(1 + |f(z)|))}{|f(z)|} \right] \frac{|f'(z)|^2}{(1 + |f(z)|)^2} d\lambda(z).$$

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