DENSE ORBITS OF RATIONALS

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ABSTRACT. Let $\mathbb{Q}$ denote the rational points of the interval $K = [0, 1)$. We construct a one-to-one piecewise linear map $\phi: K \rightarrow K$ which has the following properties:

- (1) for any $x \in K$, $\phi(x) \in \mathbb{Q}$ if and only if $x \in \mathbb{Q}$;
- (2) all the orbits $O(x) = \{\phi^i(x) | i \geq 0\}$, $x \in K$, are dense in K ;
- (3) ϕ is an automorphism of the unit circle $K = [0, 1) = \mathbb{R}/\mathbb{Z}$.

This example is motivated by a question of Friedman who was interested, because of an application to logic (Dynamic Recursion Theory), in an example of a piecewise polynomial map $\phi: K \rightarrow K$ having an orbit $O(k)$ that is dense in K and lies in $\mathbb{Q}$ (for some $k \in K$).

Let a, b be two real numbers in the open interval $(0, 1)$. Define $\phi: K \rightarrow K$, $K = [0, 1)$, by

$$\phi_{a,b}(x) = \phi(x) = \begin{cases} b + \frac{(1-b)x}{a} & \text{if } 0 \leq x < a, \\ \frac{b}{1-a}(x-a) & \text{if } a \leq x < 1. \end{cases}$$

(See Figure 1 on the next page for a graph of ϕ .)

We observe that if $b + a = 1$, then $\phi_{a,b} = T_b$ where T_b is the b -rotation ($T_b(x) = x + b \bmod 1$). It is well known that if b is irrational then T_b is uniquely ergodic and, in particular, each orbit is dense in K .

In what follows we assume that $b + a < 1$. Then $k_1 = (1-b)/a > 1$ and $k_2 = b/(1-a) < 1$ are the slopes of two linear components of the graph of $\phi_{a,b}(x)$. Let $k = k_1/k_2 > 1$ and define $\pi: K \rightarrow K$ by

$$\pi(x) = \frac{\ln[1 + (k-1)x]}{\ln k}, \quad \text{for } x \in K = [0, 1).$$

One verifies that π is an automorphism of K and that

$$\phi_{a,b}(x) = \pi^{-1}(T_c(\pi(x))) = (\pi^{-1} \circ T_c \circ \pi)(x)$$

where $c = \ln k_1 / \ln k = \ln k_1 / (\ln k_1 - \ln k_2)$ and T_c denotes the c -rotation $\phi_{1-c,c}$ of the unit circle $K = [0, 1) = \mathbb{R}/\mathbb{Z}$. Thus $\phi_{a,b}$ and T_c are conjugate

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FIGURE 1. Graph of ϕ .

(by π) and, therefore, the rotation number of the circle automorphism $\phi_{a,b}$ is c .

Thus in order for $\phi_{a,b}(x) = \phi(x)$ to satisfy the conditions (1), (2), and (3) (see the abstract), we need to have $a, b \in \mathbb{Q}$, but $c \notin \mathbb{Q}$. Note that

$$c \notin \mathbb{Q} \Leftrightarrow k_1, k_2 \text{ are multiplicatively independent}$$

(which means that $\ln k_1$ and $\ln k_2$ are linearly independent over $\mathbb{Q}$). One can take (e.g.) $a = \frac{2}{5}$ and $b = \frac{1}{5}$. Then $k_1 = 2$, $k_2 = \frac{1}{3}$ are multiplicatively independent and the map $\phi: K \rightarrow K$ defined by

$$(*) \quad \phi(x) = \begin{cases} \frac{1}{5} + 2x & \text{if } 0 \leq x < \frac{2}{5}, \\ \frac{1}{3}(x - \frac{2}{5}) & \text{if } \frac{2}{5} \leq x < 1 \end{cases}$$

is conjugate to the irrational rotation T_c with $c = \ln 2 / \ln 6$. Thus the above map $(*)$ satisfies all the conditions listed in the abstract. In particular, the rotation number of the circle automorphism ϕ is $\ln 2 / \ln 6 \notin \mathbb{Q}$.

Remark. Note that the maps $\phi_{a,b} \circ T_c$ have been considered by Michael Herman [H, §6.3]. For special choices of a , b , and c , it was shown that the unique invariant probability measure on K (under $\phi_{a,b} \circ T_c$) cannot be absolutely continuous. The author first observed (using an indirect, long argument) that the rotation number of $\phi_{a,b}$ is $c = \ln k_1 / (\ln k_1 - \ln k_2)$ and that (if $c \notin \mathbb{Q}$) the unique invariant probability measure η on K (under $\phi_{a,b}$) is absolutely continuous relative to the Lebesgue measure λ , with the Radon-Nikodym derivative $d\eta/d\lambda$ lying inbetween two positive constants. This led the author to search for a nice automorphism, π , conjugating $\phi_{a,b}$ with T_c to be accountable for this phenomenon.

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