WHEN DOES RESTRICTED CONTINUITY ON CONTINUOUS FUNCTION GRAPHS IMPLY JOINT CONTINUITY?

JOHN P. DALBEC

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ABSTRACT. We generalize a result of Luzin about real functions of real variables—that $F(x, y)$ is jointly continuous if and only if $F(x, f(x))$ is continuous for every continuous f —to functions among certain classes of topological spaces.

In [3, pp. 173–176] Luzin states and proves that a function $F: [a, b] \times [c, d] \rightarrow \mathbf{R}$ is continuous if and only if its restriction to the graph of each continuous function $f: [a, b] \rightarrow [c, d]$ and $g: [c, d] \rightarrow [a, b]$ is continuous. In this theorem, necessity is easy to show even when $[a, b]$, $[c, d]$, and $\mathbf{R}$ are replaced by arbitrary topological spaces, so we will concern ourselves with proving sufficiency. An examination of Luzin's sufficiency proof shows that he uses only the hypothesis that F is continuous when restricted to the graph of each continuous function $f: [a, b] \rightarrow [c, d]$, or equivalently that the function $F_f: [a, b] \rightarrow \mathbf{R}$ is continuous for each such f , where $F_f(x) = F(x, f(x))$. Our generalizations will use a form of this hypothesis (Theorem 2) or a slightly stronger one (Theorem 1).

We recall some definitions and results from [1] for easy reference.

A space is called Fréchet if the closure of a set includes precisely the limits of all sequences in the set. A space is called sequential if a set is closed precisely when it includes the limits of all sequences included in the set. We will use many other topological terms defined in [1, 4] without explicit mention.

Lemma 1 [1, p. 78]. *If a sequence converges to a point, all of its subsequences also converge to that point.*

Lemma 2 [1, p. 111]. *The product of two first countable spaces is first countable.*

Lemma 3 [1, p. 78]. *A first countable space is Fréchet and a Fréchet space is sequential.*

Lemma 4 [1, p. 271]. *The product of two sequential spaces, one of which is sequentially compact, is sequential.*

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Our first original lemma will be used to construct a subsequence with unique x -coordinates so that a single-valued function can be constructed through the subsequence.

Lemma 5. *Let X be a T_1 space and suppose that the sequence of (not necessarily distinct) points $\{x_n\}$ converges to x_0 , and $x_n \neq x_0$ for all n . Then $\{x_n\}$ has an infinite subsequence of distinct points $\{x'_n\}$, which converges to x_0 .*

Proof. Left to the reader. $\square$

Lemma 6. *Let X be Hausdorff, and suppose that the sequence of distinct points $\{x_n\} \subset X$ converges to x_0 and that $x_n \neq x_0$ for all n . Then there exist pairwise disjoint open neighborhoods W_n of x_n for all $n > 0$.*

Proof. Since X is Hausdorff, there exist disjoint open neighborhoods U_{ij} and U_{ji} of x_i and x_j , respectively, for all i and j such that $0 \leq i < j$. Since $\{x_n\}$ converges to x_0 and U_{0j} is an open neighborhood of x_0 for all j , there exists a number N_j for each j such that $x_n \in U_{0j}$ for all $n > N_j$. Let

$$W_n = \bigcap_{\substack{i=0 \\ i \neq n}}^{N_n} U_{ni} \cap \bigcap \{U_{0k} \mid 1 \leq k < n, x_n \in U_{0k}\}.$$

Then each W_n is open, being the intersection of finitely many open sets. We show that $W_m \cap W_n = \emptyset$ for all m and n such that $m < n$. Suppose that $n \leq N_m$. Then $W_m \subset U_{mn}$ and $W_n \subset U_{nm}$. Since $U_{mn} \cap U_{nm} = \emptyset$, $W_m \cap W_n = \emptyset$. Suppose, to the contrary, that $n > N_m$. Then $x_n \in U_{0m}$, so $W_n \subset U_{0m}$. Since $W_m \subset U_{m0}$ and $U_{0m} \cap U_{m0} = \emptyset$, $W_m \cap W_n = \emptyset$. Thus the sets W_n are indeed pairwise disjoint. $\square$

Lemma 7. *Let Y be first countable and locally path-connected. Then Y has at each point y_0 a decreasing sequence of path-connected open sets forming a local basis.*

Proof. Let $\{G_n\}$ be a countable local basis for Y at y_0 . Let $G_n^1 = \bigcap_{k=1}^n G_k$ for all n . Since Y is locally path-connected, there exist path-connected open neighborhoods G_n^2 of y_0 for each n such that $G_n^2 \subset G_n^1$. Let $G_n^3 = \bigcup_{k=n}^{\infty} G_k^2$ for all n . Then $\{G_n^3\}$ is also a countable local basis at y_0 because $G_k^2 \subset G_k^1 \subset G_n$ for all $k \geq n$ and so $G_n^3 \subset G_n$. We show that the sets G_n^3 are path-connected. Fix two points $x, z \in G_n^3$. Then there exist numbers k, m such that $x \in G_k^2$ and $z \in G_m^2$. Since $y_0 \in G_n^2$ for all n , there exist paths $p, q: I \rightarrow Y$ such that $p(0) = x$, $p(1) = q(0) = y_0$, and $q(1) = z$. Let

$$r(x) = \begin{cases} p(2x), & x \leq \frac{1}{2}, \\ q(2x - 1), & x \geq \frac{1}{2}. \end{cases}$$

Then r is continuous, $r(0) = x$, and $r(1) = z$. Finally, we show that $G_{n+1}^3 \subset G_n^3$, i.e., the sequence G_n^3 is decreasing. For all n , we have

$$G_n^3 = \bigcup_{k=n}^{\infty} G_k^2 \supset \bigcup_{k=n+1}^{\infty} G_k^2 = G_{n+1}^3.$$

Thus $\{G_n^3\}$ is the required sequence. $\square$

The first generalization of Luzin's Theorem places fewer conditions on the domain X of the functions f than the second generalization will, and no conditions at all on the range Z of F , but only at the expense of supposing that F has continuous x -sections. We note that the condition that $X \times Y$ be sequential may be satisfied either by taking X to be first countable (by Lemmas 2 and 3) or by taking X to be sequential and either X or Y to be sequentially compact (by Lemma 4).

Theorem 1. *Let X be completely regular. Let Y be first countable and locally path-connected. Let Z be a topological space. Suppose that $X \times Y$ is sequential, that the function $F: X \times Y \rightarrow Z$ has continuous x -sections, and that for any continuous function $f: X \rightarrow Y$, the function F_f is continuous. Then F is also continuous.*

Proof. Suppose that F is discontinuous. Then there is an open set $W \subset Z$ such that $F^{-1}(W)$ is not open in $X \times Y$. Since $X \times Y$ is sequential, there must be a sequence $\{(x_n, y_n)\}$ in $(X \times Y) \setminus F^{-1}(W)$ converging to a point (x_0, y_0) in $F^{-1}(W)$. But since W is an open neighborhood of $F(x_0, y_0)$ and $F(x_n, y_n) \notin W$ for all n , $\{F(x_n, y_n)\}$ cannot even accumulate to $F(x_0, y_0)$. Therefore $F(x_0, y_0)$ is not a limit point of any subsequence of $\{F(x_n, y_n)\}$.

Let $\{(x_n^1, y_n^1)\}$ be the sequence of all (x_n, y_n) such that $x_n = x_0$. Suppose that $\{(x_n^1, y_n^1)\}$ is infinite. Then since F has continuous x -sections, by Lemma 1 we have $\lim F(x_n^1, y_n^1) = \lim F(x_0, y_n^1) \ni F(x_0, y_0)$. This contradicts our earlier conclusion that $F(x_0, y_0)$ is not a limit point of any subsequence of $\{F(x_n, y_n)\}$. Thus $\{(x_n^1, y_n^1)\}$ is finite. Let $\{(x_n^2, y_n^2)\}$ be the sequence of all (x_n, y_n) such that $x_n \neq x_0$. Then $\{(x_n^2, y_n^2)\} = \{(x_n, y_n)\} \setminus \{(x_n^1, y_n^1)\}$ is infinite, so by Lemma 1 it converges to (x_0, y_0) . Then, by Lemma 5, $\{x_n^2\}$ has a subsequence of distinct points $\{x_n^3\}$. For each n choose m such that $x_n^3 = x_m^2$ and let $y_n^3 = y_m^2$. Then $\{(x_n^3, y_n^3)\}$ is a subsequence of $\{(x_n^2, y_n^2)\}$ with distinct x -coordinates.

Let $\{G_n\}$ be a decreasing sequence of path-connected open sets forming a basis for Y at y_0 as in Lemma 7. Since $\{y_n^3\}$ converges to y_0 and G_1 is an open neighborhood of y_0 , there exists a number N such that $y_n^3 \in G_1$ for all $n > N$. Let $(x_n^4, y_n^4) = (x_{n+N}^3, y_{n+N}^3)$ for all $n \geq 1$. Then $y_n^4 \in G_1$ for all n , so that $m_n = \sup\{m \mid 1 \leq m \leq n, y_n^4 \in G_m\}$ exists for each n . Since G_{m_n} is path-connected for all n , there exist paths $p_n: I \rightarrow Y$ such that $p_n(0) = y_0$, $p_n(1) = y_n^4$, and $p_n(I) \subset G_{m_n}$ for all n . Since $\{x_n^4\} \subset \{x_n^3\}$, $\{x_n^4\}$ is a sequence of distinct points and $x_n^4 \neq x_0$ for all n . Therefore, by Lemma 6, there exist pairwise disjoint open neighborhoods W_n of x_n^4 for each n . Furthermore, since X is completely regular, there exist continuous functions $f_n: X \rightarrow I$ such that $f_n(x_n^4) = 1$ and $f_n(X \setminus W_n) = 0$ for all n .

Let

$$f(x) = \begin{cases} p_n(f_n(x)), & x \in W_n, \\ y_0, & x \notin \bigcup_{n=1}^{\infty} W_n. \end{cases}$$

Let V be open in Y . If $y_0 \notin V$, then $f^{-1}(V) = \bigcup_{n=1}^{\infty} f_n^{-1}(p_n^{-1}(V))$ is open. If $y_0 \in V$, then there exists a number M such that $G_M \subset V$. Since $\{y_n^4\}$ converges to y_0 and G_M is an open neighborhood of y_0 , there exists a number N such that $y_n^4 \in G_M$ for all $n > N$. Therefore $p_n(I) \subset G_{m_n} \subset G_M \subset V$ for

each $n > \max(M, N)$, where m_n was defined above for each n . Therefore

$$f^{-1}(V) = X \setminus f^{-1}(Y \setminus V) = X \setminus \bigcup_{n=1}^N f_n^{-1}(p_n^{-1}(Y \setminus V))$$

is open, being the complement of the union of finitely many closed sets. Thus f is continuous, so

$$\lim F(x_n^4, y_n^4) = \lim F(x_n^4, f(x_n^4)) = \lim F_f(x_n^4) \ni F_f(x_0) = F(x_0, y_0).$$

This contradicts our assumption that F is discontinuous, from which we deduced that $F(x_0, y_0)$ is not a limit point of any subsequence of $F(x_n, y_n)$. $\square$

The second generalization of Luzin's Theorem dispenses with the condition that F have continuous x -sections by assuming additional hypotheses on the spaces X and Z , including the assumption that X is dense in itself. Note that one must assume something like this about X if F need not have continuous x -sections because of the trivial example $X = \{0, 1\}$, $Y = Z = [0, 1]$, $F(x, 0) = 0$, and $F(x, y) = 1$, otherwise. F is discontinuous, but F_f is continuous for every continuous $f: X \rightarrow Y$. Since much of the proof of Theorem 2 is identical to the proof of Theorem 1, we give only the additions to that proof needed to prove this theorem.

Theorem 2. *Let X be first countable, dense in itself, and completely regular. Let Y be first countable and locally path-connected. Let Z be first countable and T_3 . Given a function $F: X \times Y \rightarrow Z$, suppose that for every continuous function $f: X \rightarrow Y$ the function F_f is continuous. Then F is continuous.*

Proof. The alterations to the proof of Theorem 1 required to prove Theorem 2 follow. First, since X and Y are first countable, $X \times Y$ is first countable by Lemma 2, hence sequential by Lemma 3. Next, if the sequence $\{(x_n^1, y_n^1)\}$ is infinite, we argue as follows:

Let $\{U_n\}$ and $\{V_n\}$ be countable bases for X at x_0 and for Z at $F(x_0, y_0)$, respectively. For any open neighborhood J of $F(x_0, y_0)$, $Z \setminus J$ is a closed set not including $F(x_0, y_0)$. Since Z is T_3 , there exist disjoint open neighborhoods U and V of $F(x_0, y_0)$ and $Z \setminus J$, respectively. Since $\{V_n\}$ is a countable basis at $F(x_0, y_0)$, there exists a number m such that $V_m \subset U$. Then, since $U \subset (Z \setminus V) \subset J$ and $Z \setminus V$ is closed, the closure of V_m is a subset of $Z \setminus V$, which in turn is a subset of J . Let K_n be the closure of V_n for each n . Then for any open neighborhood J of $F(x_0, y_0)$, there exists n such that $K_n \subset J$.

Let $g_n: X \rightarrow Y$ be defined by $g_n(x) = y_n$ for each n . Then let

$$J_n = \bigcap \{U_i \mid 1 \leq i \leq n, x_n \in U_i\} \setminus F_{g_n}^{-1}(\bigcup \{K_j \mid 1 \leq j \leq n, F(x_n, y_n) \notin K_j\}),$$

where we make the convention that the intersection of the empty family of subsets of X is X itself. Note that $x_n \in J_n$ and J_n is open for each n , being the difference of a finite intersection of open sets and the continuous inverse image of a finite union of closed sets. Since X is dense in itself, this means that each J_n includes at least two points, and in particular, $J_n \setminus \{x_0\}$ is nonempty. Now for all n , choose $x_n^2 \in J_n \setminus \{x_0\}$ and let $y_n^2 = y_n$. We show that the sequence $\{x_n^2\}$ converges to x_0 . Let U be an open neighborhood of x_0 . Then

there exists m such that $U_m \subset U$. Since U_m is an open neighborhood of x_0 and $\{x_n\}$ converges to x_0 , there exists N such that $x_n \in U_m$ for all $n > N$. Then $x_n^2 \in U_m \subset U$ for all $n > \max(m, N)$ by the choice of x_n^2 . Therefore $\{x_n^2\}$ converges to x_0 , as desired.

Finally, after having shown that a subsequence of $\{F(x_n^2, y_n^2)\}$ converges to $F(x_0, y_0)$, as was done in the proof of Theorem 1, we show that the corresponding subsequence of $\{F(x_n, y_n)\}$ converges to $F(x_0, y_0)$. Let V be an open neighborhood of $F(x_0, y_0)$. Then there exists a number m such that $K_m \subset V$. If $\{F(x_n^2, y_n^2)\}$ converges to $F(x_0, y_0)$, then there exists a number N such that $F(x_n^2, y_n^2) \in K_m \subset V$ for all $n > N$. Then $F(x_n, y_n) \in K_m \subset V$ for all $n > \max(m, N)$, by the choice of x_n^2 . Therefore $\{F(x_n, y_n)\}$ converges to $F(x_0, y_0)$. $\square$

We now present two counterexamples to show the necessity for some of the hypotheses in our theorems. The first counterexample shows that the hypothesis that Y is locally path-connected cannot be omitted from either theorem. The space Y given in the example is in fact not locally connected, so this example does not rule out the possibility that local connectedness could replace local path-connectedness in the hypotheses of either theorem.

Example 1. The Perforated Unit Interval. Let $Y = I \setminus \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. Let $X = Z = I$. Then if $f: X \rightarrow Y$ is continuous, $f(X)$ is connected (and locally connected) since X is. If $0 \in f(X)$, then $f(X) = \{0\}$. Otherwise, let $y \in f(X) \setminus \{0\}$. Then there exists a number n such that $1/n < y$, and $f^{-1}([0, 1/n))$ and $f^{-1}((1/n, 1])$ are disjoint nonempty open sets whose union is X , contradicting the connectedness of I . Let

$$F(x, y) = \begin{cases} 0, & (x, y) = (0, 0), \\ xy/(x^2 + y^2), & (x, y) \neq (0, 0). \end{cases}$$

Then F is continuous except at $(0, 0)$, and F is separately continuous everywhere. Since the only continuous function through $(0, 0)$ is constant, F satisfies the hypotheses of both theorems, but is not continuous.

The second counterexample uses a space that was first described by Hewitt in [2], or rather a variant of this space that is described in [4]. This counterexample shows that F need not be continuous if all the hypotheses of Theorem 1 are satisfied except the complete regularity of X and possibly the sequentiality of $X \times Y$.¹

Example 2 [4, pp. 111–113]. Hewitt's Condensed Corkscrew. We refer the interested reader to [4] for the construction and properties of this space, which we call X here. We will use the facts that X is regular, but not completely regular, and that every continuous real-valued function on X is constant. We also need a nonrepeating convergent sequence in X . In the notation and terminology of [4], the sequence $\{(0, n)\}$ in any plank of any of the "Tychonoff Corkscrews" of X converges to the point $(0, \omega)$ in the same plank in the product topology on A , hence in the weaker condensed topology on A . Since this sequence and its limit point are included in X , the sequence converges in X .

¹As far as I know, the sequentiality of Hewitt's "Corkscrew" has not been determined [4, p. 198].

Now let $Y = I$ and let Z be Sierpiński space [4, pp. 44–46]. Let $\{x_n\}$ be a nonrepeating convergent sequence in X that does not include its limit. Define $F(x_n, 1/n) = 1$ and $F(x, y) = 0$ for all other points (x, y) of $X \times Y$. Then F is discontinuous since $\lim F(x_n, 1/n) = \lim 1 = 1$ whereas $F(\lim(x_n, 1/n)) = F(\lim(x_n), 0) = 0$. Nevertheless, we will show that F has continuous x -sections and is continuous on the graph of every continuous function from X to Y . Since $\{1\}$ is the only nontrivial closed set in Sierpiński space, a function f into Sierpiński space is continuous if and only if $f^{-1}(1)$ is closed. For any $x \in X$, let F_x be the restriction of F to $\{x\} \times Y$. Then $F_{x_n}^{-1}(1) = \{1/n\}$. Since X and Y are both T_1 , their product $X \times Y$ is T_1 also and so all singletons in $X \times Y$ are closed, in particular the singleton sets $\{(x_n, 1/n)\}$. For other values of x , $F_x^{-1}(1) = \emptyset$, which is closed in any space. Thus F has continuous x -sections. Now let f be a continuous (hence constant) function from X to Y . Then $F_f^{-1}(1) = x_n$ if f is identically $1/n$, or $F_f^{-1}(1) = \emptyset$ otherwise. Thus F_f is continuous for every continuous f .

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853
E-mail address: dalbec@math.cornell.edu