A RESULT ON MULTIDIMENSIONAL QUANTIZATION

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ABSTRACT. For any integer $N > 1$, a probability space, a Gaussian random vector X defined on the space with a positive definite covariance matrix, and an N -level quantizer Q are presented such that the random vector $Q(X)$ takes on each of the N values in its range with equal probability and such that X and $Q(X)$ are independent.

INTRODUCTION

Quantization, the process by which a set is mapped into a finite subset of a given cardinality, plays a pivotal role in virtually any application that requires analog to digital conversion; indeed, it is at the heart of much of modern digital technology. In such applications, a quantizer is often taken to be a function mapping $\mathbf{R}^k$ into a subset of $\mathbf{R}^k$ of cardinality N , where k is a positive integer and N is an integer greater than one (see, e.g., [1, 5, 3, 6, 2, 7, 9]). In this paper we present what might be a surprising consequence of such a general approach to quantization.

DEVELOPMENT

For a topological space T , we will let $\mathcal{B}(T)$ denote the family of Borel subsets of T . For a set S , we will let $\mathbf{P}(S)$ denote the power set of S and I_S denote the indicator function of S . By a standard Gaussian measure we will mean a Gaussian measure whose first moment is zero and whose second moment is one. Let k be a positive integer. For any measure m on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k))$ we will let m_* denote the inner measure on $(\mathbf{R}^k, \mathbf{P}(\mathbf{R}^k))$ induced by m and we will let m^* denote the outer measure on $(\mathbf{R}^k, \mathbf{P}(\mathbf{R}^k))$ induced by m . Recall from [4, p. 61] that if $B \in \mathcal{B}(\mathbf{R}^k)$ and $A \in \mathbf{P}(\mathbf{R}^k)$, then $m_*(B \cap A) + m^*(B \cap A^c) = m(B)$. We will let λ denote Lebesgue measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ and Λ denote Lebesgue measure on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k))$ for integers $k > 1$, where k will be determined from the context. Recall that for a measure space

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$(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k), m)$, a subset S of $\mathbf{R}^k$ is said to be a saturated non- m -measurable set if $m_*(S) = m_*(S^c) = 0$. Finally, a k -dimensional quantizer of a random variable X defined on a probability space $(\Omega, \mathcal{F}, P)$ is any function $Q: \mathbf{R}^k \rightarrow F$ such that F is a finite subset of $\mathbf{R}^k$, such that $Q(x) = x$ for all x in F (i.e., such that Q restricted to F is the identity map on F) and such that $Q(X)$ is itself a random variable defined on $(\Omega, \mathcal{F}, P)$. If F is a finite subset of $\mathbf{R}^k$ with cardinality N then a quantizer $Q: \mathbf{R}^k \rightarrow F$ of a random variable X is said to be an N -level quantizer.

The following lemma is proved in [8, pp. 381–382].

Lemma 1. *For any positive integer M there exist M disjoint subsets $Z_1, Z_2, \dots, Z_M$ of the real line such that $Z_1, Z_2, \dots, Z_M$ and $Z = Z_1 \cup \dots \cup Z_M$ are saturated non- λ -measurable sets.*

The next result is an immediate consequence of Lemma 1.

Corollary 1. *For any integer $N > 1$ there exist N subsets $T_1, T_2, \dots, T_N$ of the real line that partition the real line and are such that for each positive integer $j \leq N$, T_j is a saturated non- λ -measurable set.*

For our purposes the following corollary will prove useful.

Corollary 2. *For any positive integer k and any integer $N > 1$, there exist N subsets $S_1, S_2, \dots, S_N$ of $\mathbf{R}^k$ that partition $\mathbf{R}^k$ and are such that, for each positive integer $j \leq N$, S_j is a saturated non- Λ -measurable set.*

Proof. For $k = 1$, the result follows from Corollary 1. Assume $k > 1$. Let $T_1, \dots, T_N$ be a partition of the real line as given by Corollary 1. For positive integers $j \leq N$, let $S_j = T_j \times \mathbf{R} \times \dots \times \mathbf{R} \subset \mathbf{R}^k$. Fix a positive integer $j \leq N$ and assume that there exists an $\mathcal{F}_\sigma$ subset B of $\mathbf{R}^k$ such that $B \subset S_j$ and $\Lambda(B) > 0$. Define a subset $\hat{B}$ of $\mathbf{R}$ as follows:

$$\hat{B} = \{b_1 \in \mathbf{R} : (b_1, b_2, \dots, b_k) \in B \text{ for some } (b_2, \dots, b_k) \in \mathbf{R}^{k-1}\}.$$

Note that $\hat{B} \in \mathcal{B}(\mathbf{R})$. Further, notice that $\lambda(\hat{B}) > 0$ since $B \subset \hat{B} \times \mathbf{R} \times \dots \times \mathbf{R} \subset \mathbf{R}^k$ and $\Lambda(B) > 0$. But, $\lambda(\hat{B}) = 0$ since $\hat{B} \subset T_j$ and $\lambda_*(T_j) = 0$. This contradiction implies that $\Lambda(B) = 0$ and hence that $\Lambda_*(S_j) = 0$. It follows similarly that $\Lambda_*(S_j^c) = 0$ also. Q.E.D.

Lemma 2. *For a positive integer k and an integer $N > 1$, let $S_1, S_2, \dots, S_N$ comprise a partition of $\mathbf{R}^k$ such that for each positive integer $j \leq N$, S_j is a saturated non- Λ -measurable set. The set*

$$\mathcal{F} = \{(S_1 \cap A_1) \cup \dots \cup (S_N \cap A_N) : A_i \in \mathcal{B}(\mathbf{R}^k) \text{ for } 1 \leq i \leq N\}$$

is a σ -algebra on $\mathbf{R}^k$.

Proof. Choosing $A_1 = \dots = A_N = \emptyset$ implies that $\emptyset \in \mathcal{F}$. Let A be an element of $\mathcal{F}$. Then $A = (S_1 \cap A_1) \cup \dots \cup (S_N \cap A_N)$ for some choice of the A_i 's from $\mathcal{B}(\mathbf{R}^k)$. Further, $A^c = (S_1 \cap A_1)^c \cap \dots \cap (S_N \cap A_N)^c$. Since

$$S_i^c = \bigcup_{j=1 \atop (i \neq j)}^N S_j,$$

it follows that

$$A^c = \bigcap_{i=1}^N \bigcup_{j=1, (i \neq j)}^N S_j \cup A_i^c.$$

Hence A^c is a finite union of sets, each of which is of one of the following three forms:

- (i) $S_{n_1} \cap \cdots \cap S_{n_k} \cap B$ where $1 \leq n_1 < \cdots < n_k \leq N$, $k > 1$, and $B \in \mathcal{B}(\mathbf{R}^k)$;
- (ii) $S_j \cap B$ for $1 \leq j \leq N$ and $B \in \mathcal{B}(\mathbf{R}^k)$;
- (iii) $B \in \mathcal{B}(\mathbf{R}^k)$.

Every set of the form given by (i) is empty since the S_i 's are disjoint. Further, any set $B \in \mathcal{B}(\mathbf{R}^k)$ may be expressed as $B = (S_1 \cap B) \cup \cdots \cup (S_N \cap B)$. Hence, A^c is an element of $\mathcal{F}$.

Finally, if $B_1, B_2, \dots$ are in $\mathcal{F}$, then for some choice of the $A_{i,j}$'s from $\mathcal{B}(\mathbf{R}^k)$,

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^N (S_j \cap A_{i,j}) = \bigcap_{j=1}^N S_j \cap \left(\bigcup_{i=1}^{\infty} A_{i,j} \right) \in \mathcal{F}. \quad \text{Q.E.D.}$$

Recall that two measures P_1 and P_2 on a given measurable space $(\Omega, \mathcal{G})$ are said to be equivalent if $\{A \in \mathcal{G} : P_1(A) = 0\} = \{A \in \mathcal{G} : P_2(A) = 0\}$. Notice that for sets $S_1, S_2, \dots, S_N$ as above, it follows that, for any positive integer $i \leq N$ and any $\mathcal{B}(\mathbf{R}^k)$ -measurable set H , $P_*(S_i \cap H) = 0$, $P_*(S_i^c \cap H) = 0$, $P^*(S_i \cap H) = P(H)$, and $P^*(S_i^c \cap H) = P(H)$ for any probability measure P on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k))$ that is equivalent to Lebesgue measure on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k))$. The following lemma will be used in the proof of a subsequent theorem.

Lemma 3. *For a positive integer k and an integer $N > 1$, let $S_1, S_2, \dots, S_N$ comprise a partition of $\mathbf{R}^k$ such that for each positive integer $j \leq N$, S_j is a saturated non- Λ -measurable set. Let P be a probability measure on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k))$ that is equivalent to Lebesgue measure on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k))$. Let $A_1, \dots, A_N$ and $B_1, \dots, B_N$ be sets from $\mathcal{B}(\mathbf{R}^k)$ such that*

$$(S_1 \cap A_1) \cup \cdots \cup (S_N \cap A_N) = (S_1 \cap B_1) \cup \cdots \cup (S_N \cap B_N).$$

Then $P(A_i \Delta B_i) = 0$ for any positive integer $i \leq N$ where for any two subsets A and B of $\mathbf{R}^k$, $A \Delta B$ denotes the symmetric difference of A and B .

Proof. Fix a positive integer $i \leq N$. By assumption,

$$(S_1 \cap A_1) \cup \cdots \cup (S_N \cap A_N) = (S_1 \cap B_1) \cup \cdots \cup (S_N \cap B_N).$$

Intersecting each side with S_i implies that $(S_i \cap A_i) = (S_i \cap B_i)$, which implies that $(S_i \cap A_i) \cap (S_i \cap B_i)^c = (S_i \cap A_i) \cap (S_i^c \cup B_i^c) = (S_i \cap A_i \cap S_i^c) \cup (S_i \cap A_i \cap B_i^c) = (S_i \cap A_i \cap B_i^c) = \emptyset$ and, similarly, that $(S_i \cap B_i \cap A_i^c) = \emptyset$. Thus, we see that $(S_i \cap A_i \cap B_i^c) \cup (S_i \cap B_i \cap A_i^c) = S_i \cap (A_i \Delta B_i) = \emptyset$. Since $(A_i \Delta B_i) \in \mathcal{B}(\mathbf{R}^k)$, it follows that $P(A_i \Delta B_i) = P^*(S_i \cap (A_i \Delta B_i)) = P^*(\emptyset) = 0$. Q.E.D.

The following theorem provides a probability space upon which the principal result of this paper will be based.

Theorem 1. *For a positive integer k and an integer $N > 1$, let $S_1, S_2, \dots, S_N$ comprise a partition of $\mathbf{R}^k$ such that for each positive integer $j \leq N$, S_j is a saturated non- Λ -measurable set. Let P be a probability measure on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k))$*

that is equivalent to Lebesgue measure on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k))$. There exists a probability space $(\mathbf{R}^k, \mathcal{G}, \mu)$ such that $\mathcal{G}$ includes $\mathcal{B}(\mathbf{R}^k)$, such that $\mathcal{G}$ contains $S_1, \dots, S_N$, such that the measure μ agrees with P on $\mathcal{B}(\mathbf{R}^k)$, and such that $\mathcal{B}(\mathbf{R}^k)$ is independent of $\sigma(S_1, \dots, S_N)$.

Proof. Let $\mathcal{G}$ be the σ -algebra $\mathcal{F}$ provided by Lemma 2. Recall that $\mathcal{F}$ contains all sets of the form $(S_1 \cap A_1) \cup \dots \cup (S_N \cap A_N)$ where $A_i \in \mathcal{B}(\mathbf{R}^k)$ for each positive integer $i \leq N$. If $A \in \mathcal{B}(\mathbf{R}^k)$ then choosing $A_1 = \dots = A_N = A$ implies that $A \in \mathcal{G}$. Similarly, for any positive integer $i \leq N$, setting $A_i = \mathbf{R}^k$ and all other A_j 's equal to the empty set implies that $S_i \in \mathcal{G}$. Define a measure μ on the measurable space $(\mathbf{R}^k, \mathcal{G})$ via

$$\mu((S_1 \cap A_1) \cup \dots \cup (S_N \cap A_N)) = \frac{1}{N}(P(A_1) + \dots + P(A_N))$$

for $(S_1 \cap A_1) \cup \dots \cup (S_N \cap A_N) \in \mathcal{G}$. That μ is well defined follows from Lemma 3 and that μ is in fact a probability measure that agrees with P on $\mathcal{B}(\mathbf{R}^k)$ is then straightforward. Further notice that $\mu(S_i) = 1/N$ for each positive integer $i \leq N$ and that, for any set $B \in \mathcal{B}(\mathbf{R}^k)$ and any positive integer $i \leq N$, $\mu(S_i \cap B) = \frac{1}{N}P(B) = \mu(S_i)\mu(B)$. Thus S_i is independent of $\mathcal{B}(\mathbf{R}^k)$ for each positive integer $i \leq N$. Finally, notice that $\mathcal{B}(\mathbf{R}^k)$ is in fact independent of $\sigma(S_1, \dots, S_N)$ since $\{\emptyset, S_1, \dots, S_N\}$ is a π -system. Q.E.D.

We are now in a position to state and prove the principal result of this paper.

Theorem 2. *Let k be a positive integer and let N be an integer greater than one. There exists a probability space $(\Omega, \mathcal{S}, \nu)$, a Gaussian random vector X defined on $(\Omega, \mathcal{S}, \nu)$ taking values in $\mathbf{R}^k$ with a positive definite covariance matrix, and an N -level k -dimensional quantizer $Q: \mathbf{R}^k \rightarrow F$ such that $\nu(Q(X) = x) = 1/N$ for each x in F and such that X and $Q(X)$ are independent.*

Proof. Let $S_1, \dots, S_N$ be sets as provided by Corollary 2. For these N subsets of $\mathbf{R}^k$, let $(\Omega, \mathcal{S}, \nu)$ be a probability space as provided by Theorem 1 where P is chosen to be the product measure induced by placing standard Gaussian measure on each factor of $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k))$. For each positive integer $i \leq N$, let α_i be an element from S_i . Let F denote the set $\{\alpha_1, \dots, \alpha_N\}$. Define an N -level k -dimensional quantizer $Q: \mathbf{R}^k \rightarrow F$ via $Q(x) = \sum_{i=1}^N \alpha_i I_{S_i}(x)$. Further, notice that the random vector $X(\omega) = \omega$; $\omega \in \Omega$, is a zero mean Gaussian random vector defined on $(\Omega, \mathcal{S}, \nu)$ whose covariance matrix is the $k \times k$ identity matrix. Also, notice that for $1 \leq i \leq N$, $\nu(Q(X(\omega)) = \alpha_i) = \nu(\omega \in S_i) = 1/N$. Finally, notice that X and $Q(X)$ are independent via Theorem 1. Q.E.D.

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