THE BAIRE CATEGORY AND FORCING LARGE LINDELÖF SPACES WITH POINTS G_δ

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ABSTRACT. For κ as large an aleph as we want, we construct by forcing a model in which CH holds and there is a Lindelöf zero-dimensional space of size κ with points G_δ .

In 1922 (published 1929) Alexandroff and Urysohn proved, for Hausdorff Lindelöf spaces X , that if closed sets are G_δ then $|X| \leq \mathfrak{c}$. They asked whether it is sufficient to assume only that X has countable character [1].

In 1969 Arhangel'skii showed that this is indeed the case [2]. And he, in turn, asked whether countable pseudo-character (points are G_δ sets) is enough.

In 1978 Shelah gave a consistent answer *NO* to the question of Arhangel'skii by constructing a model of CH and in it a Lindelöf space with points G_δ of size $\mathfrak{c}^+ = \aleph_2$ [3, 4]. This was the best result up to this point. The question of the true limit of the size of Lindelöf spaces with points G_δ was still unanswered. In particular, it *was* an open problem to find, consistent with CH, such a space of size $\aleph_3$ [4].

In this paper we build for any cardinal λ a model in which $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} > \lambda$ and there is a Lindelöf, 0-dimensional Hausdorff space, with points G_δ and cardinality $2^{\aleph_1}$. The problem of whether consistently there can be such spaces of size larger than $2^{\aleph_1}$ remains open.

Assume, without loss of generality, that $V \models \text{CH}$ & $\kappa^{\aleph_1} = \kappa$. Fix a disjoint partition of κ into countable infinite sets and its enumeration $\{A_\alpha : \alpha < \kappa\}$, and, for each α in κ , a countable set H_α dense in 2^{A_α} , $H_\alpha = \{h_\xi : \xi \in A_\alpha\} \subset 2^{A_\alpha}$.

Notation. For $F \subset 2^\kappa$ and $K \subset \kappa$, let $F \restriction K := \{f \restriction K : f \in F\}$. For $s \in \text{Fn}(\kappa, 2, \kappa^+)$, let $U_s := [s] := \{f \in 2^\kappa : s \subset f\}$. Hence, for example, $U_s \restriction K = \{f \restriction K : s \subset f\} \subset 2^K$. If $B \subset \text{Fn}(A, 2)$, let $U_B := \bigcup \{U_s : s \in B\}$.

Let $p \in \mathbb{P}$ iff $p = \langle F, T \rangle$ and there are (necessarily unique) I, A, G satisfying the following conditions (and then we also write $\langle I, A, G, F, T \rangle = \langle I^p, A^p, G^p, F^p, T^p \rangle$):

- (1) $I \in [\kappa]^{\leq \omega}$;
- (2) $A = \bigcup \{A_\alpha : \alpha \in I\}$;
- (3) $G = \{g_\alpha : \alpha \in I\}$ and $(\forall \alpha \in I) g_\alpha : A \setminus A_\alpha \rightarrow 2$;

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(4) $F = \{f_\xi : \xi \in A\} \subset 2^A$ and $(\forall \alpha \in I)(\forall \xi \in A_\alpha) (f_\xi \upharpoonright (A \setminus A_\alpha) = g_\alpha \text{ and } f_\xi \upharpoonright A_\alpha = h_\xi)$.

Remark. So $f_\xi = g_\alpha \cup h_\xi$ and $(\forall \alpha \in I) \{f_\xi : \xi \in A_\alpha\}$ is a *dense* subset of $U_{g_\alpha} \upharpoonright A$. The set $U_{g_\alpha} \upharpoonright A$ is a closed G_δ (hence nowhere dense, if $|I| > 1$) in 2^A , but of second category in itself, because it is isomorphic to the Cantor set 2^{A_α} .

(5) $(\forall \alpha \in I)(\forall \beta \in I \setminus \{\alpha\})(\forall \xi \in A_\alpha) g_\beta \upharpoonright A_\alpha \neq h_\xi$.

Remark. Therefore $(\forall \alpha \in I)(\forall \xi \in A_\alpha)(\forall \eta \in A \setminus \{\xi\}) f_\xi \upharpoonright A_\alpha \neq f_\eta \upharpoonright A_\alpha$, and so A_α witnesses that $\{f_\xi\}$ will be a G_δ , for every ξ in A_α .

(6) $|T| \leq \omega$ and $(\forall B \in T) B \subset Fn(A, 2)$ with $F \subset U_B \upharpoonright A := \bigcup \{U_s \upharpoonright A : s \in B\}$, i.e., $(\forall \xi \in A)(\exists s \in B)$ with $s \subset f_\xi$.

Remark. In other words, B codes an open cover $U_B \upharpoonright A$ of $F \subset 2^A$. Hence, by (4), $(\forall B \in T)(\forall \alpha \in I) U_B \upharpoonright A$ is dense open in $U_{g_\alpha} \upharpoonright A$. We say $q \leq p$ in $\mathbb{P}$ iff

- (a) $I^q \supset I^p$,
- (b) $(\forall \xi \in A^p) f_\xi^q \upharpoonright A^p = f_\xi^p$, and
- (c) $T^q \supset T^p$.

Note. (b) is equivalent to $(\forall \alpha \in I^p) g_\alpha^q \upharpoonright A^p = g_\alpha^p$.

Fact 1. $(\forall p \in \mathbb{P})(\forall \alpha \in \kappa)(\exists q \leq p)$ with $\alpha \in I^q$. Hence $(\forall \xi \in \kappa) D_\xi := \{q \in \mathbb{P} : \xi \in A^q\}$ is dense in $\mathbb{P}$.

Fact 2. $\mathbb{P}$ is countably closed, moreover, ω_1 -complete.

Fact 3. $\mathbb{P}$ has the ω_2 -chain condition.

Definition. Let G be a $\mathbb{P}$ -generic filter over V . For every $\xi \in \kappa$, let $f_\xi := \bigcup \{f_\xi^p : p \in G \text{ \& } \xi \in A^p\}$. Let $F := \{f_\xi : \xi \in \kappa\}$

Fact 4. In $V[G]$, $2^\omega = \omega_1$, $2^{\omega_1} = \kappa$, all cardinals are preserved. Also $(\forall \xi \in \kappa) f_\xi : \kappa \rightarrow 2$ is a total function and F is a points G_δ subspace of 2^κ , in particular, $|F| = \kappa$.

Fact 5. In $V[G]$, F is Lindelöf.

The proof of the facts will involve no combinatorics.

Fact 1 is true by the Baire Category Theorem. Without loss of generality, $\alpha \notin I^p$.

If $I^p = \emptyset$, let $q := \langle H_\alpha, \emptyset \rangle$. Otherwise, pick $\beta \in I^p$ and find, by the Baire Category Theorem,

$$g_\alpha \in (U_{g_\beta} \upharpoonright A^p) \cap \bigcap \{U_B \upharpoonright A^p : B \in T^p\} \setminus \{f_\xi : \xi \in A_\beta\} \neq \emptyset.$$

Also pick $h_* \in 2^{A_\alpha} \setminus H_\alpha$. Now let $I^q := I^p \cup \{\alpha\}$ and, for every $\gamma \in I^p$, set $g_\gamma^q := g_\gamma^p \cup h_*$, and let $G^q := \{g_\gamma^q : \gamma \in I^p\} \cup \{g_\alpha\}$ and $T^q := T^p$.

Fact 2 is obvious.

Fact 3 is true by the Baire Category Theorem: Suppose $\mathbb{Q} \subset \mathbb{P}$ and $|\mathbb{Q}| = \omega_2$. We will show that we can find two conditions, p and q , in $\mathbb{Q}$ that can be extended to some $r \leq p, q$ in $\mathbb{P}$.

By CH, the Δ -system lemma and simple counting we may assume that there are $p \neq q$ in $\mathbb{Q}$ that are isomorphic structures, i.e., that $\text{tp } I^p = \text{tp } I^q$, $\text{tp } A^p =$

$\text{tp } A^q$, and denoting by ξ' the unique ordinal in A^q corresponding to a $\xi \in A^p$ such that $\text{tp}(A^p \cap \xi) = \text{tp}(A^q \cap \xi')$, we have $(\forall \xi \in A^p) (\forall \eta \in A^p) f_\xi^p(\eta) = f_{\xi'}^q(\eta')$, and, moreover, $\forall x \in \Delta := A^p \cap A^q (= \bigcup \{A_\alpha : \alpha \in I^p \cap I^q\}) f_\xi^p(x) = f_{\xi'}^q(x)$.

We shall construct an $r \in \mathbb{P}$ with $r \leq p, q$ of the form $r = \langle I^p \cup I^q, A^p \cup A^q, F^r, G^r, T^p \cup T^q \rangle$. For every $\xi \in A^p$, f_ξ^r will be an extension of f_ξ^p on $A^q \setminus \Delta$, and for every $\xi' \in A^q$, $f_{\xi'}^r$ will extend $f_{\xi'}^q$ on $A^p \setminus \Delta$. Of course, we need to extend, appropriately, only g_α 's.

There are three cases.

Case 1. $\alpha \in I^p \cup I^q$. Set $g_\alpha^r := g_\alpha^p \cup g_\alpha^q$.

Case 2. $\alpha \in I^p \setminus I^q$. By the Baire Category Theorem, there is an $h : A^q \rightarrow 2$ with

$$h \in (U_{g_\alpha} \upharpoonright A^q) \cap \bigcap \{U_B \upharpoonright A^q : B \in T^q\} \setminus \{f_\eta^q : \eta \in A_{\alpha'}\} \neq \emptyset.$$

Then let $g_\alpha^r := g_\alpha^p \cup h$.

Case 3. $\alpha' \in I^q \setminus I^p$. This is the same as Case 2. There is an $h : A^p \rightarrow 2$ with

$$h \in (U_{g_\alpha} \upharpoonright A^p) \cap \bigcap \{U_B \upharpoonright A^p : B \in T^p\} \setminus \{f_\eta^p : \eta \in A_\alpha\}.$$

Set $g_{\alpha'}^r := g_{\alpha'}^q \cup h$. That r thus defined belongs to $\mathbb{P}$ (and, hence, extends both, p and q) is immediate.

All of Fact 4, except $\kappa \leq 2^{\omega_1}$, follows from Facts 1, 2, and 3; and $\kappa \leq 2^{\omega_1}$ follows, for example, from the observation that $\{g_\alpha \upharpoonright \omega_1 : A_\alpha \cap \omega_1 = \emptyset\}$ is an enumeration of distinct functions on ω_1 .

Proof of Fact 5. First observe that if $p \in \mathbb{P}$ and $B \in T^p$, then $p \Vdash \text{“}\check{U}_B \text{ covers } \check{F}\text{”}$. Indeed, for every $\xi \in \kappa$, $D(B, \xi) := \{q \in \mathbb{P} : q \Vdash \check{F}(\xi) \in \check{U}_B\}$ is dense below p by Fact 1 (and because $\{q \in \mathbb{P} : q \leq p \ \& \ \xi \in A^q\} \subset D(B, \xi)$).

Let σ be a $\mathbb{P}$ -name for a basic open cover (in 2^κ) of $\check{F}$, and let some p force this, i.e., $p \Vdash \text{“}\sigma : \check{\kappa} \longrightarrow \check{F}n(\kappa, 2) \ \& \ ((\forall \xi \in \kappa) \sigma(\xi) \subset \check{F}(\xi))\text{”}$.

It is enough to show that there is an extension $q \leq p$ that decides the values of $\sigma(\xi)$ for all $\xi \in A^q$ and such that the domains of these decided functions $\sigma(\xi)$ all lie inside A^q .

That is, we want $q \leq p$ and $B = \{s_\xi : \xi \in A^q\} \subset Fn(A^q, 2)$ with $q \Vdash \text{“}(\forall \xi \in A^q) \sigma(\xi) = \check{s}_\xi\text{”}$. Then we may let $q^* := \langle F^q, T^q \cup \{B\} \rangle$ with $q \geq q^* \in \mathbb{P}$. And then, by our observation above, $q^* \Vdash \text{“}\sigma \upharpoonright \check{A}^{q^*} = \check{B} \text{ and covers } \check{F}\text{”}$.

Thus we would establish the density of such conditions below a member of G , hence the existence of such q^* in G , hence our Fact. But such $q \leq p$ exists because $\mathbb{P}$ is ω_1 -complete.

To wit, $\forall p \leq p$ and $\forall \xi \in A^p \exists r \leq p$ that decides $\sigma(\xi)$ (since σ is mapping into V), so $\exists s_\xi \in Fn(\kappa, 2)$ with $r \Vdash \sigma(\xi) = \check{s}_\xi$; and, by Fact 1, we may assume that $s_\xi \subset A^r$. Then, “closing up” and using Fact 2, we may see that there is a $q \leq p$ we want.

Remarks. 1. As $\mathbb{P}$ stands, there are $p \in \mathbb{P}$ which force $\check{F}$ not to be dense in 2^κ . For example, a $p \in \mathbb{P}$ with $I^p = \{0, 1\}$ and

$$T^p = \{\{\{\xi_0, g_1(\xi_0)\}\}, \{\{\xi_1, g_0(\xi_1)\}\}\},$$

where ξ_0 = an ordinal in A_0 and ξ_1 = some ordinal in A_1 . Then $p \Vdash \text{“}\check{U}_{\{\langle \xi_0, 1-g_1(\xi_0) \rangle, \langle \xi_1, 1-g_0(\xi_1) \rangle\}} \cap \check{F} = \emptyset\text{”}$.

2. But with a slightly more work, we can make $\dot{F}$ dense in 2^κ , and hence, e.g., c.c.c. For example, we could additionally require that, for $p \in \mathbb{P}$, F^p be dense in 2^{A^p} ; or, alternatively, that for every B in T^p , $U_B \upharpoonright A^p$ be dense in 2^{A^p} (and then the density of $\dot{F}$ would follow by genericity).

3. An unessential modification of $\mathbb{P}$ will yield in the extension $V[G']$ two Lindelöf spaces (with points G_δ) F and F' such that their product $F \times F'$ has a closed discrete subspace of size $\kappa = 2^{\aleph_1}$. As a consequence, in $V[G']$, $2^{\aleph_1}$ is “subbase Lindelöf.” Indeed, let $H'_\alpha := \{h'_\xi : \xi \in A_\alpha\}$, where $h'_\xi \upharpoonright (A_\alpha \setminus \{\xi\}) := h_\xi \upharpoonright (A_\alpha \setminus \{\xi\})$ and $h'_\xi(\xi) := 1 - h_\xi(\xi)$. Define, for $\xi \in A_\alpha$, $f'_\xi := g_\alpha \cup h'_\alpha$, $F' := \{f'_\xi : \xi \in A\}$ and force with (the conditions containing) two sets of covers, T of F and T' of F' , simultaneously. This works, because H'_α again is dense in 2^{A_α} . Then, in the generic extension, $\{\langle f_\xi, f'_\xi \rangle : \xi < \kappa\}$ is a closed discrete subspace of $F \times F'$. And since its two projections are Lindelöf, it is “subbase Lindelöf.”

4. If $\aleph_1 < \lambda < \kappa$ with $\text{cf}(\lambda) > \omega$, then adding λ Cohen reals to $V[G]$ (i.e., forcing from $V[G]$ with $\text{Fn}(\lambda, 2)$) we are getting a model in which $2^{\aleph_0} = \lambda > \aleph_1$ and our space F is still Lindelöf (see, e.g., [5, Lemma 3.3]). This answers another consistency question.

5. The Lindelöf property of F can be destroyed by a simple c.c.c. forcing. Indeed, take any $\alpha \in \kappa$ such that $A_\alpha \cap \omega_1 = \emptyset$. Then $\forall \gamma < \omega_1$

$$D_\gamma := \{p \in \mathbb{P} : (\exists \xi \in \kappa \setminus A_\alpha) p \Vdash \dot{f}_\xi \upharpoonright \gamma = \dot{g}_\alpha \upharpoonright \gamma\}$$

is dense in $\mathbb{P}$ (use the proof of the Fact 1). Hence in $V[G]$ there is a set of size ω_1

$$S = \{f_{\xi_\gamma} : \gamma < \omega_1\} \text{ s.t. } \forall \gamma < \omega_1 \ f_{\xi_\gamma} \upharpoonright \gamma = g_\alpha \upharpoonright \gamma.$$

If $f \in F$ is a complete accumulation point (CAP) of S , then $f \upharpoonright \omega_1 = g_\alpha \upharpoonright \omega_1$. Therefore, $f \in \{f_\xi : \xi \in A_\alpha\}$. Let C = the set of all CAPs of S , then $\bar{C} = C \subset \{f_\xi : \xi \in A_\alpha\}$. Therefore $|C| \leq \aleph_0$ and $\Psi(C, F) = \aleph_1$.

Now force the Martin's Axiom MA_{ω_1} from our $V[G]$ as a ground model. Extending an argument in [6], we have $MA_{\omega_1} \Rightarrow \mathfrak{b} > \aleph_1 \Rightarrow$ “Countable closed subsets in a Lindelöf space with points G_δ cannot have pseudocharacter exactly $\aleph_1$.” (In general, $\Psi(C, F) \notin [\aleph_1, \mathfrak{b})$.) Therefore, in the resulting model of MA_{ω_1} , S has a subset of size $\aleph_1$ without a CAP in F , and so F is not Lindelöf (it has an open cover of size $\aleph_1$ without a subcover of a smaller size).

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