SHIMURA SUBGROUPS OF JACOBIANS OF SHIMURA CURVES

SAN LING

(Communicated by William W. Adams)

ABSTRACT. Given an indefinite quaternion algebra of reduced discriminant D and an integer N relatively prime to D , one can construct Shimura curves $\text{Sh}_0(N, D)$ and $\text{Sh}_1(N, D)$, which are analogues of $X_0(N)$ and $X_1(N)$. The natural morphism $\text{Sh}_1(N, D) \rightarrow \text{Sh}_0(N, D)$ induces a morphism $J_0(N, D) \rightarrow J_1(N, D)$ between the Jacobians. We compute the kernel $\Sigma(N, D)$ of this latter map, which is finite.

Let $\mathcal{B}$ be an indefinite quaternion algebra over $\mathbf{Q}$ that is ramified at an even number of primes $l_1, \dots, l_n$. Then $\mathcal{B}$ has reduced discriminant $D = \prod_i l_i$. Let $N \geq 1$ be an integer that is relatively prime to D . Suppose that p is a prime dividing N .

The algebra $\mathcal{B}_p \stackrel{\text{def}}{=} \mathcal{B} \otimes \mathbf{Q}_p$ is isomorphic to the matrix algebra $M_2(\mathbf{Q}_p)$. Let

$$\mathcal{O}_{p,m} = \begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p^m \mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix}$$

be the canonical Eichler order of level $p^m \mathbf{Z}_p$ in $M_2(\mathbf{Q}_p)$. We define the groups $\Gamma_0(p^m, \mathbf{Z}_p)$ and $\Gamma_1(p^m, \mathbf{Z}_p)$ to be

$$\begin{aligned} \Gamma_0(p^m, \mathbf{Z}_p) &= \text{SL}_2(\mathbf{Z}_p) \cap \mathcal{O}_{p,m}, \\ \Gamma_1(p^m, \mathbf{Z}_p) &= \left\{ x \in \Gamma_0(p^m, \mathbf{Z}_p) \mid x \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{p^m \mathcal{O}_{p,0}} \right\}. \end{aligned}$$

Let $\mathcal{O}$ be a $\mathbf{Z}$ -Eichler order in $\mathcal{B}$ of level N , and let p^m be the exact power of p dividing N . For such a p , let $i_p: \mathbf{Q} \hookrightarrow \mathbf{Q}_p$ be the canonical embedding. This embedding can be extended to an embedding, also denoted by i_p , of $\mathcal{B}$ in $M_2(\mathbf{Q}_p)$, such that $i_p(\mathcal{O})$ is the canonical Eichler order $\mathcal{O}_{p,m}$ of level $p^m \mathbf{Z}_p$ [4, pp. 108–109]. The inverse image of $\Gamma_0(p^m, \mathbf{Z}_p)$ under i_p is $\mathcal{O}^1$, the set of elements of $\mathcal{O}$ of (reduced) norm 1. We define the congruence subgroups $\Gamma_0(N, D)$ and $\Gamma_1(N, D)$ as

$$\begin{aligned} (1) \quad \Gamma_0(N, D) &= \{x \in \mathcal{O}^1 \mid i_p(x) \in \Gamma_0(p^m, \mathbf{Z}_p) \text{ for all } p \mid N\}, \\ \Gamma_1(N, D) &= \{x \in \mathcal{O}^1 \mid i_p(x) \in \Gamma_1(p^m, \mathbf{Z}_p) \text{ for all } p \mid N\}. \end{aligned}$$

Received by the editors October 16, 1991.

1991 *Mathematics Subject Classification.* Primary 11G18.

This research is financially supported by the National University of Singapore Overseas Graduate Scholarship.

Since $\mathcal{B}$ is unramified at the infinite place of $\mathbf{Q}$, it follows that $\mathcal{B} \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to $M_2(\mathbf{R})$. Upon fixing such an isomorphism, we obtain an embedding $i_\infty : \mathcal{O}^1 \hookrightarrow \mathrm{SL}_2(\mathbf{R})$. The action of $\mathrm{SL}_2(\mathbf{R})$ on the Poincaré upper half plane $\mathcal{H} = \{\tau \in \mathbf{C} \mid \mathrm{Im}(\tau) > 0\}$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto \frac{a\tau + b}{c\tau + d}$$

implies that the groups $\Gamma_i(N, D)$ ($i = 0, 1$) also act on $\mathcal{H}$. When $D > 1$, the quotient $\{\pm 1\}\Gamma_i(N, D)\backslash\mathcal{H}$ ($i = 0, 1$) has a natural structure of compact connected Riemann surface. It is called a *Shimura curve*, which we shall denote by $\mathrm{Sh}_i(N, D)$. (When $D = 1$, the quaternion algebra $\mathcal{B}$ is the matrix algebra $M_2(\mathbf{Q})$. The quotients $\{\pm 1\}\Gamma_i(N, 1)\backslash\mathcal{H}$ ($i = 0, 1$) are known to be noncompact. Their compactifications are the classical modular curves $X_i(N)$ ($i = 0, 1$) described in the introduction of [2].)

From the inclusion $\Gamma_1(N, D) \subseteq \Gamma_0(N, D)$, we get a natural morphism of curves $u : \mathrm{Sh}_1(N, D) \rightarrow \mathrm{Sh}_0(N, D)$.

Let $J_i(N, D)$ denote the Jacobian of $\mathrm{Sh}_i(N, D)$. Let

$$u^* : J_0(N, D) \rightarrow J_1(N, D)$$

be the morphism of abelian varieties deduced from u via Pic functoriality. Its kernel, called the *Shimura subgroup*, is a finite group; we denote it by $\Sigma(N, D)$.

The Shimura subgroup $\Sigma(N, D)$ when $D = 1$ (i.e., in the case of classical modular curves) is studied in [3] when N is prime and in [2] for arbitrary N . In this note, we study $\Sigma(N, D)$ for $D > 1$.

As in the case of classical modular curves, we see readily that the morphism $u : \mathrm{Sh}_1(N, D) \rightarrow \mathrm{Sh}_0(N, D)$ is a Galois covering with Galois group $\Gamma_0(N, D)/\{\pm 1\}\Gamma_1(N, D)$. We can identify $\Gamma_0(N, D)/\Gamma_1(N, D)$ with $(\mathbf{Z}/N\mathbf{Z})^\times$ as follows.

For $x \in \Gamma_0(N, D)$, let

$$i_p(x) = \begin{pmatrix} a_p(x) & b_p(x) \\ c_p(x) & d_p(x) \end{pmatrix} \in \Gamma_0(p^m, \mathbf{Z}_p).$$

Consider the reduction $d_p(x) \in (\mathbf{Z}_p/p^m\mathbf{Z}_p)^\times \simeq (\mathbf{Z}/p^m\mathbf{Z})^\times$. By the Chinese Remainder Theorem, there exists a unique $d(x) \in (\mathbf{Z}/N\mathbf{Z})^\times$ such that $d(x) \equiv d_p(x) \pmod{p^m}$ for all $p|n$. The homomorphism $d : \Gamma_0(N, D) \rightarrow (\mathbf{Z}/N\mathbf{Z})^\times = \prod (\mathbf{Z}/p^m\mathbf{Z})^\times$, given by $x \mapsto d(x)$, is surjective and clearly has kernel $\Gamma_1(N, D)$. Hence,

$$(2) \quad \Gamma_0(N, D)/\Gamma_1(N, D) \simeq (\mathbf{Z}/N\mathbf{Z})^\times.$$

Let $\mathbf{U}$ be the group of complex numbers of modulus 1. We prove in §1 the main theorem of this paper.

Theorem 1. *The Shimura subgroup $\Sigma(N, D)$ of $J_0(N, D)$ is canonically isomorphic to the group of homomorphisms $g : (\mathbf{Z}/N\mathbf{Z})^\times \rightarrow \mathbf{U}$ such that $g(d) = 1$*

if $d = -1$ or $d^2 + td + 1 = 0$, where

$$t \in \begin{cases} \{0, 1\} & \text{if, for all } l|D, (-4/l) \neq 1 \neq (-3/l), \\ \{0\} & \text{if, for all } l|D, (-4/l) \neq 1, \\ & \text{but there exists } l|D \text{ such that } (-3/l) = 1, \\ \{1\} & \text{if, for all } l|D, (-3/l) \neq 1, \\ & \text{but there exists } l|D \text{ such that } (-4/l) = 1, \\ \emptyset & \text{if there exist } l|D \text{ and } l'|D \text{ such that } (-3/l) = 1 = (-4/l'). \end{cases}$$

Corollary 1. Let $D > 1$, let $\phi(N)$ denote the number of elements of $(\mathbf{Z}/N\mathbf{Z})^\times$, and:

- (i) let k be the number of prime divisors of N distinct from 2 and 3;
- (ii) let m_2 be equal to 2 if, for all l dividing D , $(-4/l) \neq 1$, and -1 is a square mod N (i.e., if $4 \nmid N$ and each prime factor $p \neq 2$ of N is congruent to 1 mod 4), and let m_2 be equal to 1 otherwise;
- (iii) let m_3 be equal to 3 if, for all l dividing D , $(-3/l) \neq 1$, and $X^2 + X + 1$ has a root mod N (i.e., if $9 \nmid N$ and each prime factor $p \neq 3$ of N is congruent to 1 mod 3), and let m_3 be equal to 1 otherwise.

Then we have

$$\text{Card}(\Sigma(N, D)) = \begin{cases} \phi(N)/(2m_2^k m_3^k) & \text{if } N \geq 5, \\ 1 & \text{if } N \leq 4. \end{cases}$$

Corollary 2. Let $D > 1$, let $N = \prod p^{r_p}$ be the prime power decomposition of N , and:

- (i) let r'_p be equal to $r_p - 1$ if $p \neq 2$;
- (ii) let r'_2 be equal to $\max(0, r_2 - 2)$;
- (iii) let e_0 be equal to $\text{lcm}_{p|N}((p-1)p^{r'_p})$;
- (iv) let m_1 be equal to 2 if N is the product of 1 or 2 by a power of an odd prime, and let m_1 be equal to 1 otherwise;
- (v) let m_2 and m_3 be as in Corollary 1.

Then the exponent of the group $\Sigma(N, D)$ (i.e., the smallest integer e such that $e\Sigma(N, D) = 0$) is given by

$$e = \begin{cases} e_0/(m_1 m_2 m_3) & \text{if } N \geq 5, \\ 1 & \text{if } N \leq 4. \end{cases}$$

Corollary 3. For $D > 1$, the only integers N for which the order of $\Sigma(N, D)$ is 1 are

$$\begin{cases} 1, 2, 3, 4, 6 & \text{for any } D > 1, \\ 5, 10 & \text{if, for all } l|D, (-4/l) \neq 1, \\ 7 & \text{if, for all } l|D, (-3/l) \neq 1, \\ 13 & \text{if, for all } l|D, (-3/l) \neq 1 \neq (-4/l). \end{cases}$$

Corollary 4. When N approaches infinity, the exponent and a fortiori the order of $\Sigma(N, D)$ go to infinity.

1. PROOF OF THEOREM 1

Let Γ be a discrete subgroup of $\text{SL}_2(\mathbf{R})$ of finite covolume. Suppose that the Riemann surface $X = \Gamma \backslash \mathcal{H}$ is compact and let $J(X)$ be the Jacobian. Then

the argument in [2, §1.5] produces a canonical injective group homomorphism

$$(3) \quad \psi_{\Gamma} : J(X) \rightarrow \text{Hom}(\Gamma, \mathbf{U}),$$

where $\mathbf{U}$ is the group of complex numbers of modulus 1.

The groups $\Gamma_0(N, D)$ and $\Gamma_1(N, D)$ can be regarded as discrete subgroups of $\text{SL}_2(\mathbf{R})$ (via i_{∞}) of finite covolume. With $\Gamma = \Gamma_0(N, D)$, we obtain from (3) the injection

$$\psi : J_0(N, D) \rightarrow \text{Hom}(\Gamma_0(N, D), \mathbf{U}),$$

and that an element $x \in J_0(N, D)$ belongs to the Shimura subgroup $\Sigma(N, D)$ if and only if the kernel of $\psi(x)$ contains $\Gamma_1(N, D)$. Using the identification (2), we deduce from ψ a canonical injection

$$(4) \quad \psi' : \Sigma(N, D) \rightarrow \text{Hom}((\mathbf{Z}/N\mathbf{Z})^{\times}, \mathbf{U}).$$

Furthermore, a homomorphism $h : \Gamma_0(N, D) \rightarrow \mathbf{U}$ belongs to the image of ψ if and only if its kernel contains all the elliptic elements of $\Gamma_0(N, D)$.

From [4, IV 3A], the elliptic elements of $\Gamma_0(N, D)$ can only be of order 2 or 3, i.e., they satisfy either $x^2 + 1 = 0$ or $x^2 + x + 1 = 0$. There are elliptic elements of order 2 (resp. 3) if and only if $(-4/l) \neq 1$ (resp. $(-3/l) \neq 1$) for all $l|D$. Consequently, a homomorphism $g : (\mathbf{Z}/N\mathbf{Z})^{\times} \rightarrow \mathbf{U}$ belongs to the image of ψ' if and only if its kernel contains S' , where S' contains -1 and the roots in $\mathbf{Z}/N\mathbf{Z}$ of the polynomials $X^2 + tX + 1 = 0$ for

$$t \in \begin{cases} \{0, 1\} & \text{if, for all } l|D, (-4/l) \neq 1 \neq (-3/l), \\ \{0\} & \text{if, for all } l|D, (-4/l) \neq 1, \\ & \text{but there exists } l|D \text{ such that } (-3/l) = 1, \\ \{1\} & \text{if, for all } l|D, (-3/l) \neq 1, \\ & \text{but there exists } l|D \text{ such that } (-4/l) = 1, \\ \emptyset & \text{if there exist } l|D \text{ and } l'|D \text{ such that } (-3/l) = 1 = (-4/l'). \end{cases}$$

This proves Theorem 1.

2. ORDERS AND EXPONENTS OF $\Sigma(N, D)$

2.1. Order of $\Sigma(N, D)$. Let $\mu_2(\mathbf{Z}/N\mathbf{Z})$ and $\mu_3(\mathbf{Z}/N\mathbf{Z})$ denote the groups of square roots and cube roots of unity in $\mathbf{Z}/N\mathbf{Z}$, respectively. We make use of the two following lemmas to determine the order of $\Sigma(N, D)$.

Lemma 1. Assume that there exists $a \in \mathbf{Z}/N\mathbf{Z}$ such that $a^2 + 1 = 0$. The subgroup of $(\mathbf{Z}/N\mathbf{Z})^{\times}$ generated by the roots of $X^2 + 1$ is equal to $\mu_2(\mathbf{Z}/N\mathbf{Z}) \cup \mu_2(\mathbf{Z}/N\mathbf{Z})a$; it contains -1 . The order of this group is 2^{k+1} (where k is as in Corollary 1).

Lemma 2. Assume that $X^2 + X + 1$ has a root in $\mathbf{Z}/N\mathbf{Z}$. The subgroup of $(\mathbf{Z}/N\mathbf{Z})^{\times}$ generated by these roots is then equal to $\mu_3(\mathbf{Z}/N\mathbf{Z})$. The order of this group is 3^k .

The above are Lemmas 2 and 3 in [2]. The proofs are given there.

Corollary 1 now follows immediately from Theorem 1 and Lemmas 1 and 2.

2.2. **Exponent of $\Sigma(N, D)$.** Let e be the exponent of $\Sigma(N, D)$ and let $N \geq 5$. We prove Corollary 2 by considering the two following cases:

(a) *When $m_2 = 2$.* In this case, N is of the form N' or $2N'$, with $N' \neq 1$ and each prime factor of N' congruent to 1 mod $4m_3$, and $\Sigma(N, D)$ is isomorphic to the quotient of the group

$$A = \prod_{p|N'} \left(\left(\mathbb{Z}/\frac{p-1}{2m_3}\mathbb{Z} \right) \times (\mathbb{Z}/p^{r'_p}\mathbb{Z}) \right)$$

by the unique subgroup of order 2 that has a nonzero projection on each factor $\mathbb{Z}/((p-1)/2m_3)\mathbb{Z}$. If N' has at least two distinct prime factors ($m_1 = 1$), then e is equal to e_A , the exponent of A . If N' is a prime power ($m_1 = 2$), then $e = e_A/2$. Consequently,

$$e = \frac{e_A}{m_1} = \frac{e_0}{m_1 \cdot 2m_3} = \frac{e_0}{m_1 m_2 m_3}.$$

(b) *When $m_2 = 1$.* In this case, the subgroup $\Sigma(N, D)$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times / \{\pm 1\} \mu_{m_3}(\mathbb{Z}/N\mathbb{Z})$. If $m_3 = 1$, then $e = e_0/m_1$ since the exponent of $(\mathbb{Z}/N\mathbb{Z})^\times / \{\pm 1\}$ is e_0/m_1 . If $m_3 = 3$, then $2 \nmid N$ and an argument similar to the one in (a) shows that $e = e_0/3m_1 = e_0/m_1 m_3$. Therefore, we have $e = e_0/m_1 m_2 m_3$.

2.3. **The case where $e = 1$.** For each odd prime divisor p of N , the exponent r_p of p in the prime power decomposition of N is at most 1; otherwise, p would divide e . Moreover, $p-1$ divides $m_1 m_2 m_3 e$ and a fortiori 12, hence $p \in \{3, 5, 7, 13\}$. Furthermore, the exponent r_2 of 2 is at most 2; otherwise 2 would divide e .

The integer N cannot be divisible by $2 \cdot 7$, $2 \cdot 13$, $5 \cdot 7$, or $5 \cdot 13$ since $m_3 = 1$ and $3|e$ in these cases. It is not divisible by $3 \cdot 5$, $3 \cdot 7$, $3 \cdot 13$, $7 \cdot 13$, $2^2 \cdot 3$, or $2^2 \cdot 5$ since $2|e$ in these cases. Hence, N can only be 1–7, 10, and 13.

For $1 \leq N \leq 4$ and $N = 6$, we have $\Sigma(N, D) = \{1\}$ from Corollary 1.

When $N = 5$, we have $e_0 = 4$, $m_1 = 2$, and $m_3 = 1$. We need $m_2 = 2$ to ensure $e = 1$. This means that $(-4/l) \neq 1$ for all l dividing D .

When $N = 7$, we have $e_0 = 6$, $m_1 = 2$, and $m_2 = 1$. In order to have $e = 1$, we need $m_3 = 3$, which is the case provided $(-3/l) \neq 1$ for all l dividing D .

When $N = 10$, we have $e_0 = 4$, $m_1 = 2$, and $m_3 = 1$. To have $e = 1$, we require $m_2 = 2$, which is true when $(-4/l) \neq 1$ for all l dividing D .

When $N = 13$, we have $e_0 = 12$ and $m_1 = 2$. We need $m_2 = 2$ and $m_3 = 3$ so that $e = 1$. This means that $(-4/l) \neq 1$ and $(-3/l) \neq 1$ for all l dividing D .

This completes the proof of Corollary 3.

2.4. **Behaviour of exponent when N approaches infinity.** It is clear from Corollary 2 that if the exponent of $\Sigma(N, D)$ is bounded then the prime divisors of N are bounded and so are the exponents of the prime divisors. Hence, the set of positive integers N for which the exponent of $\Sigma(N, D)$ is bounded is finite. This means that e tends to infinity as N approaches infinity.

ACKNOWLEDGMENT

The author wishes to thank Ken Ribet for helpful conversations while working on this project.

REFERENCES

1. B. Jordan, *On the diophantine arithmetic of Shimura curves*, Ph.D. thesis, Harvard University, 1981.
2. S. Ling and J. Oesterlé, *The Shimura subgroup of $J_0(N)$* , *Astérisque* **196–197** (1991), 171–203.
3. B. Mazur, *Modular curves and the Eisenstein ideal*, *Inst. Hautes. Études. Sci. Publ. Math.* **47** (1978), 33–186.
4. M.-F. Vignéras, *Arithmétique des algèbres de quaternions*, *Lecture Notes in Math.*, vol. 800, Springer-Verlag, Berlin and New York, 1980.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, National University of Singapore, Singapore 0511, Singapore

E-mail address: matlings@nuscc.nus.sg