THE STRUCTURE OF SOME RIGHT NOETHERIAN RINGS WITH KRULL DIMENSION ONE

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ABSTRACT. In this note we prove a structure theorem for a right Noetherian ring R with krull dimension one having a right Artinian quotient ring and, moreover, that has a finitely generated, faithful, critical right module U with krull dimension U equal to one. We end this note with some examples that clarify certain features of this theorem.

NOTATION AND TERMINOLOGY

Let R be a ring and M a right R -module. We denote the right krull dimension of M or R by $|M|$ or $|R|$. For basic properties of krull dimension, the reader is referred to [6]. A module with krull dimension is said to be α -krull homogeneous or simply krull homogeneous in the case where $|M| = \alpha$, if $|N| = |M|$, for every nonzero submodule N of M . The ring R is said to be krull homogeneous if the module R_R is krull homogeneous. For the definition of Primary module or Primary ring and various results about Prime ideals associated to a module we refer the reader to Gordon [4]. If A is a ring then its nilradical will be denoted by $N(A)$. The set of regular elements of A is denoted by $C(O)$ and $C(B)$ denotes the set of elements of A that are regular modulo B where B is an ideal of A . If S_i , $i \in I$, are subrings of a ring R then their internal direct sum is denoted by $\dot{+}_{i \in I} S_i$. A ring R is said to be nonsingular if the right singular ideal of R is zero. The set of associated prime ideals of a right module M is denoted by $\text{Ass}(M)$. As usual rings are with identity and modules are unitary.

MAIN THEOREM

Proposition 1. *Let R be a right Noetherian ring with M a faithful, finitely generated, critical module over R such that $|M| = |R|$. Then R is a krull homogeneous P -primary ring, where P is the associated prime of M . Furthermore, if I is an essential right ideal of R then $|R/I| < |R|$.*

Proof. For the proof of this proposition see Gordon [5, Lemma (1.7)] and Boyle and Feller [1, Corollary (2.5)].

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Proposition 2. *Let R be a nonsingular, right Artinian ring that is P -primary for some prime ideal P of R . Then $P = eR$ for some idempotent e of R .*

Proof. If $P = 0$, then $e = 0$ and so there is nothing to prove. So, suppose $P \neq 0$. Let I be a right ideal maximal with the property that $IP = 0$. I is clearly a two-sided ideal of R and P is a maximal ideal of R because R is right Artinian. Notice that by Gordon [4], I is a large right ideal of R . So $I \not\subseteq P$ because R is nonsingular. Also $P \not\subseteq I$ for otherwise $P = I$, and hence P would be large. Hence $P + I = R$. Now one can easily verify that $P = eR$ for some nontrivial idempotent e of R .

Theorem 3 (Main theorem). *Let R be a right Noetherian ring with $|R| = 1$ and such that R has a right Artinian quotient ring Q . If U is a finitely generated, faithful, critical module over R with $|U| = |R|$, and if $\text{Ass } U = P$ for some prime ideal P of R , then the following hold.*

- (i) $P = eR$ for some idempotent e of R .
- (ii) $R = S + m + T$, where S is a right Artinian ring that is nonsingular and primary for some Prime ideal of S , m is an ideal of R with $m^2 = 0$, and T is a nonzero prime ring such that $T \approx R/P$ with $|T| = 1$.
- (iii) $R = R_1 + N(R)$ where R_1 is a Semiprime ring with $R_1 \approx R/N(R)$.

Proof. If $P = 0$ then there is nothing to prove. So we suppose that $P \neq 0$.

(i) By Proposition 1 R is krull homogeneous, P -Primary. Also R is nonsingular. Let I be a right ideal maximal with respect to the property $IP = 0$. I is clearly a two-sided ideal of R that is also a large right ideal by Gordon [4]. Hence by Proposition 1 R/I is a right Artinian ring. Let $c \in C(0)$. Then $cr \in I \Rightarrow crP = 0$ and hence $rP = 0$, which implies $r \in I$. So $c + I$ is a right regular element of R/I . Since R/I is right Artinian, $c + I$ is a unit of R/I . Thus, if Q_i , $1 \leq i \leq m$, are the Prime ideals of R that are minimal over I , then $P \not\subseteq Q_i$. For if $P \subseteq Q_i$ then Q_i/N is a nonminimal prime ideal of R/N (N being the nilradical of R), and so Q_i/N must contain a regular element of R/N . Since R has a right Artinian quotient ring, so by Small's theorem [8] we get that Q_i contains a regular element of R . By what we have done about this shows Q_i/I is not a proper ideal of R/I . Hence we must have $P \not\subseteq Q_i$. Now we claim $I + P = R$. For if $I + P \neq R$, then there exists a maximal ideal X of R such that $I + P \subseteq X$. Since R/I is a right Artinian ring, so $X/I = Q_j/I$ for some j , $1 \leq j \leq m$. Hence $I + P \subseteq X$ implies $P \subseteq X$. But $X = Q_j$ shows $P \subseteq Q_j$, contradicting our above assertion that $P \not\subseteq Q_j$. Hence $I + P = R$. Combining this with $IP = 0$, we get that there exists an $e \in R$, with $e^2 = e$ and such that $eR = eP = P$.

(ii) See Chatters and Hajarnavis [3, Lemma 8.8, 6.13] for the proof of (ii)). More explicitly, $S = eRe$, $m = eR(1 - e)$, and $T = (1 - e)R(1 - e)$. If I is as in (i) above then $I = R(1 - e)$ and $S \approx R/I$, which implies that S is a right Artinian ring.

(iii) R is nonsingular implies that S is nonsingular. Also US is a finitely generated, faithful module over S that is also a uniform S module because S is nonsingular. Thus S is X -Primary for some prime ideal X of S . Since S is a right Artinian ring, by Proposition 2 X is generated as a right ideal of S by an idempotent of S . It is easy to show by a Noetherian Induction and Chinese remainder theorem as in (i) that $S = S_1 + N(S)$ where S_1 is a semiprime ring

and where $N(S) = \text{nilradical of } S$. Also $S/N(S) \approx S_1$. Now by (ii) either

$$R = S \dot{+} m \dot{+} T \quad \text{or} \quad R = S_1 \dot{+} N(S) \dot{+} m \dot{+} T$$

or

$$R = (S_1 \dot{+} T) \dot{+} (N(S) \dot{+} m).$$

If $N = \text{nilradical of } R$, then it is easy to see that $R/N(R) \approx S_1 \dot{+} T$. So if $R_1 = S_1 \dot{+} T$ then R_1 is a semiprime ring and $N(S) \dot{+} m = N$. Thus $R = R_1 \dot{+} N$ where $R_1 \approx R/N$.

Remark. Because of Theorem 3(i), the ring R can be put isomorphic to a ring A where $A = \begin{bmatrix} S & m \\ 0 & T \end{bmatrix}$, where S is a right Artinian subring of R , m is an ideal of R with $m^2 = 0$, and $T \approx R/P$ is a nonzero prime factor ring of R .

We now end this note with some examples. Throughout here we assume F is a field and $F[x]$ the usual polynomial ring.

Example 1. There exists a ring R satisfying the conditions of the main theorem and yet R is not a prime ring.

Consider

$$R = \begin{bmatrix} F & F[x] \\ 0 & F[x] \end{bmatrix}.$$

Notice that R has a finitely generated, faithful, critical, right module

$$M = \begin{bmatrix} F & F[x] \\ 0 & 0 \end{bmatrix}$$

with $|M| = |R| = 1$. Clearly R is not a prime ring and R is right Noetherian but not left Noetherian.

Example 2. There exists a ring R_1 that is right Noetherian of krull dimension one and, moreover, that has a finitely generated, faithful, critical module M with $|M| = |R_1|$ and yet R_1 does not have an Artinian quotient ring. Let

$$R_1 = \left\{ \begin{bmatrix} a & g(x) \\ 0 & a_+(x) \end{bmatrix} \mid \begin{matrix} g(x) \in F[x], \ a \in F, \\ (x) = cx \mid c \in F[x] \end{matrix} \right\} \subseteq R$$

where R is as in Example 1. Notice that $|R_1| = 1$. Also

$$M = \begin{bmatrix} F & F[x] \\ 0 & 0 \end{bmatrix}$$

is a finitely generated, faithful, critical module over R_1 with $|M| = 1$. Notice that

$$P = \left\{ \begin{bmatrix} 0 & g(x) \\ 0 & 0 \end{bmatrix} \mid g(x) \in F[x] \right\}$$

is a prime ideal of R_1 that is associated to R_1 as well as to M . Let

$$Q = \left\{ \begin{bmatrix} 0 & g(x) \\ 0 & (x) \end{bmatrix} \mid \begin{matrix} g(x) \in F[x], \\ (x) = cx \mid c \in F[x] \end{matrix} \right\}.$$

Then Q is a maximal ideal of R_1 with $P \not\subseteq Q$. Also $QP = 0$. clearly R_1 cannot have an Artinian quotient ring.

Example 3. One may be tempted to think that S is a prime ring in the main theorem (Theorem 3). We give an example to show that S is not a prime ring. Observe first that if S is a prime ring then there would exist in R a minimal prime ideal Q ($\neq P$) such that $QP = 0$ showing, thereby, that $\min \text{Spec}(R) = \{Q, P\}$, where by $\min \text{Spec}(R)$ we mean the set of minimal prime ideals of R .

However, we will give an example of a ring A satisfying all the conditions of Theorem 3 whose $\min \text{Spec}(A)$ contains more than two prime ideals. Let

$$R = \begin{bmatrix} F & F[x] \\ 0 & F[x] \end{bmatrix}, \quad P = \begin{bmatrix} F & F[x] \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & F[x] \\ 0 & F[x] \end{bmatrix}.$$

Clearly P is a critical right ideal of R that is finitely generated and faithful over R with $|P| = |R| = 1$. Also $\min \text{Spec}(R) = \{P, Q\}$. Notice that R has a right Artinian quotient ring. Consider $A = \begin{bmatrix} F & P \\ 0 & R \end{bmatrix}$. Then A satisfies all the conditions of Theorem 3 with $U = \begin{bmatrix} F & P \\ 0 & 0 \end{bmatrix}$ is a finitely generated, faithful, critical module over A with $|U| = |A| = 1$. Notice that

$$\min \text{Spec} = \left\{ \begin{bmatrix} F & P \\ 0 & P \end{bmatrix}, \begin{bmatrix} F & P \\ 0 & Q \end{bmatrix}, \begin{bmatrix} 0 & P \\ 0 & R \end{bmatrix} \right\}.$$

Hence there are more than two elements in $\min \text{Spec} A$.

Remarks. We remark that it is an open question whether or not A is a Prime ring if A is supposed to be a Noetherian ring and, moreover, that has a finitely generated, faithful, critical right module M . This question is equivalent to the question of whether or not the nilradical of A is weakly ideal invariant. For this question and related results see [2, 7]. We mention that the above question is open even when we assume that A has an Artinian quotient ring. In view of this it becomes desirable to prove our main theorem, namely, Theorem 3 for the ring R of this theorem, when R is assumed to have krull dimension greater than one. However, we are unable to prove this result.

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