A FIXED POINT THEOREM IN NON-ARCHIMEDEAN VECTOR SPACES

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ABSTRACT. A mapping T defined on a normed linear space X and taking values in X is said to be contractive (nonexpansive) if whenever x and y are distinct points in X , $\|Tx - Ty\| < \|x - y\|$ ($\|Tx - Ty\| \leq \|x - y\|$). In this paper we prove that every contractive mapping on a spherically complete non-Archimedean normed space has a unique fixed point.

1. INTRODUCTION

A non-Archimedean normed space $(X, \|\cdot\|)$ is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection. If $T: X \rightarrow X$, then T is said to be contractive (nonexpansive) if whenever x and y are distinct points in X ,

$$\|Tx - Ty\| < \|x - y\| \quad (\|Tx - Ty\| \leq \|x - y\|).$$

For the Euclidean space $\mathbb{R}$ with its usual metric, it is known that a contractive mapping $f: \mathbb{R} \rightarrow \mathbb{R}$ need not have a fixed point, e.g., the mapping

$$f(x) = x + \frac{1}{1 + e^x}, \quad x \in \mathbb{R}.$$

In this paper we prove that every contractive mapping $T: X \rightarrow X$, where X is a spherically complete non-Archimedean normed space, has a unique fixed point. We give also examples to show that this assertion cannot be extended to include either nonexpansive mappings or nonspherically complete spaces.

It is an open problem if a nonexpansive mapping on a reflexive normed space from a convex closed and bounded set into itself has a fixed point [1–3]. From our examples it follows that an analogous problem for the case of non-Archimedean normed spaces has a negative answer.

More information about non-Archimedean normed spaces can be found in [4].

2. MAIN RESULT

Theorem 1. *Let X be a non-Archimedean spherically complete normed space. If $T: X \rightarrow X$ is a contractive mapping then T has a unique fixed point.*

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Proof. Let $B_a =: B[a, \|a - Ta\|]$ denote the closed spheres centered at a with radii $\|a - Ta\|$, and let $\mathcal{A}$ be the collection of these spheres for all $a \in X$. The relation

$$B_a \leq B_b \quad \text{iff} \quad B_b \subseteq B_a$$

is a partial order. Consider a totally ordered subfamily $\mathcal{A}_1$ of $\mathcal{A}$. From the spherical completeness of X , we have

$$\bigcap_{B_a \in \mathcal{A}_1} B_a =: B \neq \emptyset.$$

Let $b \in B$ and $B_a \in \mathcal{A}_1$. Then if $x \in B_b$,

$$\|x - a\| \leq \max\{\|a - b\|, \|b - x\|\} \leq \|a - Ta\|,$$

since $\|a - b\| \leq \|a - Ta\|$, and

$$\|x - b\| \leq \|b - Tb\| \leq \max\{\|b - a\|, \|a - Ta\|, \|Ta - Tb\|\} = \|a - Ta\|.$$

So $x \in B_a$ and $B_b \subseteq B_a$ for every $B_a \in \mathcal{A}_1$. Thus B_b is an upper bound in $\mathcal{A}$ for the family $\mathcal{A}_1$. By Zorn's lemma, $\mathcal{A}$ has a maximal element, say B_z , for some $z \in Z$. We claim that $z = Tz$.

Suppose $z \neq Tz$. Since

$$(*) \quad \|Tz - T^2z\| < \|z - Tz\| \quad (T^2 =: T \circ T)$$

and

$$Tz \in B[Tz, \|Tz - T^2z\|] \cap B[z, \|z - Tz\|],$$

we have $B_{Tz} \subseteq B_z$. But $z \notin B_{Tz}$ by $(*)$, so $B_{Tz} \subsetneq B_z$, and this contradicts the maximality of B_z . Therefore T has a fixed point, obviously unique.

3. NONEXPANSIVE MAPPINGS

Theorem 2. Suppose X is a spherically complete non-Archimedean normed space and $T: X \rightarrow X$ is a nonexpansive map.

Then either T has at least one fixed point or there exists a sphere B of radius $r > 0$ such that $T: B \rightarrow B$ and for which $\|b - Tb\| = r$ for each $b \in B$.

Proof. Defining B_a and $\mathcal{A}$ as in the proof of Theorem 1, we find similarly a maximal element B_z of $\mathcal{A}$. For any $b \in B_z$ we have

$$\|b - Tb\| \leq \max\{\|b - z\|, \|z - Tz\|, \|Tz - Tb\|\} = \|z - Tz\|,$$

thus $B_b \subseteq B_z$ (since b is a common point of B_b and B_z) and $Tb \in B_z$.

Now, if $\|b - Tb\| < \|z - Tz\|$ then

$$\|b - z\| = \|z - Tz\| > \|b - Tb\|.$$

This implies that $z \in B_z$ but $z \notin B_b$, which is impossible from the maximality of B_z . Thus

$$\|b - Tb\| = \|z - Tz\| =: r \quad \forall b \in B_z.$$

Remark. Evidently the above results hold also in the case of a spherically complete ultrametric space.

4. EXAMPLES

1. Let X be the space c_0 over $\mathcal{K}$ with the valuation of $\mathcal{K}$ discrete. It is well known that in this case c_0 is spherically complete. Let $\pi \in \mathcal{K}$ with

$0 < |\pi| < 1$. Suppose $T: c_0 \rightarrow c_0$ is the mapping defined by

$$T(x_1, x_2, x_3, \dots) = (\pi, x_1, x_2, x_3, \dots).$$

Clearly $\|Tx - Ty\| \leq \|x - y\|$, but T has no fixed point in c_0 .

2. Let X be the space c_0 over $\mathcal{K}$ with the valuation of $\mathcal{K}$ dense. As it is known, c_0 in this case is not spherically complete. Suppose T is the mapping $T: c_0 \rightarrow c_0$ defined by

$$T(x_1, x_2, x_3, \dots) = (1, s_1x_1, s_2x_2, \dots, s_nx_n, \dots),$$

where $\{s_n\}$ is a sequence in $\mathcal{K}$ with $|s_n| < 1$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} |s_n| = 1$, and $\lim_{n \rightarrow \infty} \prod_{i=1}^n |s_i| \neq 0$. The mapping T is clearly a contraction but it has no fixed points.

3. Let X be the space c_0 with $\mathcal{K}$ nonspherically complete. As is known, c_0 in this case is reflexive. The mapping of example 1 maps the unit sphere of c_0 into itself but does not have a fixed point.

So in non-Archimedean spaces, a nonexpansive mapping on a reflexive space, which maps a convex closed and bounded set into itself, does not necessarily have a fixed point.

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