A COUNTEREXAMPLE TO ORLIK'S CONJECTURE

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(Communicated by Jeffrey N. Kahn)

ABSTRACT. We present a counterexample to the conjecture by Orlik that the restriction of a free hyperplane arrangement to one of its hyperplanes is free.

I. DEFINITIONS AND THE COUNTEREXAMPLE

This note presents a counterexample to a conjecture by Orlik on hyperplane arrangements. In what follows, we give only those definitions necessary to state the conjecture and the counterexample. For more background on the theory of hyperplane arrangements, see [Or].

Let $\mathcal{H}$ be a finite set of hyperplanes (subspaces of codimension one) passing through the origin in $\mathbb{R}^d$, and for each hyperplane H in $\mathcal{H}$, let l_H be the linear form in the polynomial ring $S = \mathbb{R}[x_1, \dots, x_d]$ that vanishes on H (so that l_H is uniquely defined up to a scalar multiple). The *module of $\mathcal{H}$ -derivations* $\text{Der}(\mathcal{H})$ is defined to be the set of all derivations $\theta : S \rightarrow S$ with the property that $\theta(l_H)$ is divisible by l_H for all H in $\mathcal{H}$. $\text{Der}(\mathcal{H})$ is a module over the polynomial ring S , and we say $\mathcal{H}$ is a *free arrangement* if it is a free module over S . Given any hyperplane H in $\mathcal{H}$, we define the *restriction arrangement* $\mathcal{H}|_H$ to be the arrangement within the subspace H (thinking of H as $\mathbb{R}^{d-1}$) whose hyperplanes are all of the intersections of hyperplanes of $\mathcal{H}$ with H .

Conjecture (Orlik [T, Problem 2; Or, p. 86]). *For any hyperplane H in a free arrangement $\mathcal{H}$, the restriction arrangement $\mathcal{H}|_H$ is free.*

A counterexample to this conjecture is given by the arrangement $\mathcal{H}$ consisting of the 21 hyperplanes in $\mathbb{R}^5$ defined by the equations $x_i = 0$ for $i = 1, 2, 3, 4, 5$ and

$$x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4 + \varepsilon_5 x_5 = 0 \quad \text{where } \varepsilon_i = \pm 1 \text{ for each } i.$$

Using the computer algebra package MACAULAY, it was checked that $\text{Der}(\mathcal{H})$ is a free S -module with an S -basis consisting of five homogeneous derivations of degrees $(0, 4, 4, 4, 4)$, respectively. These derivations were then double-checked to satisfy *Saito's criterion* for freeness [Or, Theorem 9.7] using MATHEMATICA. Choosing H to be the hyperplane defined by $l_H =$

Received by the editors September 24, 1991 and, in revised form, November 6, 1991.
 1991 *Mathematics Subject Classification*. Primary 52B30.

$x_1 - x_2 - x_3 - x_4 - x_5 = 0$, one can check that $\mathcal{H}|_H$ is equivalent (up to a projective transformation, which does not affect the freeness [Z2, Proposition 2.3]) to the arrangement of 15 hyperplanes in $\mathbb{R}^4$ defined by the equations

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0$$

where $\alpha_i = 0$ or 1 for each i , and not all of the α_i are 0. A MACAULAY calculation then shows that $\text{Der}(\mathcal{H})$ is *not* a free S -module. Alternatively, a theorem of Terao says that when $\mathcal{H}$ is free, its *characteristic polynomial* $\chi(\mathcal{H}, t)$ factors into linear factors over the integers (see [Or, Theorem 9.19]), while the characteristic polynomial of $\mathcal{H}|_H$ above can be calculated by hand to be $\chi(\mathcal{H}|_H, t) = (t-1)(t-4)(t^2-10t+26)$.

II. REMARKS

(1) This counterexample was discovered while the authors were investigating the following classes of arrangements: For a positive integer n and a subset $S \subseteq \{1, 2, \dots, n\}$ consider the two sets of vectors

$$A(n, S) = \{v = (v_1, \dots, v_n) \in \mathbb{R}^n \mid v_i = 0 \text{ or } 1, \ 1 \leq i \leq n, \text{ and } \text{supp}(v) \in S\}$$

and

$$B(n, S) = \{v = (v_1, \dots, v_n) \in \mathbb{R}^n \mid v_i = 0 \text{ or } \pm 1, \ 1 \leq i \leq n, \text{ and } \text{supp}(v) \in S\}$$

where by $\text{supp}(v)$ we mean the number of nonzero coordinates of v . For example, $B(n, \{1, 2\})$ is just the usual root system B_n . The arrangement corresponding to the set $B(n, \{1, \dots, n\})$ arises in the study of the number of different permutations of the vertices of the n -cube that can arise by ordering them relative to a linear functional. The above counterexample is $B(5, \{1, 5\})$, and its restriction that is not free is $A(4, \{1, 2, 3, 4\})$.

(2) $\mathcal{H}$ also leads to a counterexample to a conjecture of Ziegler on multiarrangements, which implies Orlik's conjecture. In [Z1] he defines a *multiarrangement* $\widetilde{\mathcal{H}}$ to be an arrangement $\mathcal{H}$ along with a positive integer multiplicity m_H for every hyperplane H in $\mathcal{H}$. He also defines the notion of a *free* multiarrangement and shows how the restriction arrangement $\mathcal{H}|_H$ has a naturally associated multiarrangement $\widetilde{\mathcal{H}}|_H$. After proving that $\widetilde{\mathcal{H}}|_H$ is free when $\mathcal{H}$ is free, he conjectures that if a multiarrangement $\widetilde{\mathcal{H}}$ is free, then its underlying arrangement $\mathcal{H}$ is free [Z1, Conjecture 13]. It is easy to check that when $\mathcal{H}$ and H are as in the above counterexample, the multiarrangement $\widetilde{\mathcal{H}}|_H$ has multiplicity two on the hyperplanes of the form $x_i = 0$ and on the hyperplane $x_1 + x_2 + x_3 + x_4 = 0$, and multiplicity one on all other hyperplanes. Since $\mathcal{H}$ is free, we know that $\widetilde{\mathcal{H}}|_H$ is free, but we have seen that $\mathcal{H}|_H$ is *not* free.

(3) The matroid of $\mathcal{H}$ *cannot* be a counterexample to Terao's conjecture, as it is *projectively unique* (see [Z2, §2 and Proposition 2.3]).

(4) Although $\mathcal{H}$ is free, one can readily check that it is not *inductively free* [Or, Definition 9.16]. We do not know if Orlik's conjecture is true for inductively free arrangements.

(5) We have no reason to suspect that $\mathcal{H}$ is a minimal counterexample to Orlik's conjecture, in any sense of the word "minimal".

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