BORDISM CLASSES OF VECTOR BUNDLES OVER REAL PROJECTIVE SPACES

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ABSTRACT. A basis is presented for the $\mathbf{RP}^n$ classes in $\mathfrak{N}_n(BO)$. This basis is used to prove that every smooth involution (M, T) on a closed manifold bounds if its fixed point set is a disjoint union of odd-dimensional projective spaces of constant dimension.

Given any real vector bundle $\xi^c \rightarrow \mathbf{RP}^n$, let $(\mathbf{RP}^n, \xi) \in \mathfrak{N}_n(BO)$ denote the stable bordism class determined by the map $\mathbf{RP}^n \rightarrow BO(c) \hookrightarrow BO$ classifying ξ . With n fixed, the classes $(\mathbf{RP}^n, \xi)$ as ξ ranges over all vector bundles span a $\mathbf{Z}_2$ -vector subspace of $\mathfrak{N}_n(BO)$, where the sum $(\mathbf{RP}^n, \xi) + (\mathbf{RP}^n, \eta)$ represents the disjoint union. We will exhibit a basis for this subspace.

Letting $\lambda \rightarrow \mathbf{RP}^n$ denote the canonical line bundle, the KO-theory of real vector bundles tells us that any $\xi \rightarrow \mathbf{RP}^n$ is stably isomorphic to $k\lambda$ for some (possibly large) k , so that $(\mathbf{RP}^n, \xi) = (\mathbf{RP}^n, k\lambda)$. The main result of this paper is that up to bordism one need only consider those $k \leq n$.

Theorem. *A basis for the $\mathbf{RP}^n$ classes in $\mathfrak{N}_n(BO)$ is*

$$\begin{aligned} \{(\mathbf{RP}^n, j\lambda) \mid 0 \leq j \leq n\} & \quad \text{if } n = 2m, \\ \{(\mathbf{RP}^n, (2j+1)\lambda) \mid 0 \leq j \leq m\} & \quad \text{if } n = 2m+1. \end{aligned}$$

Thus, given a bundle $k\lambda \rightarrow \mathbf{RP}^n$ with $k > n$, the theorem tells us that

$$(\mathbf{RP}^n, k\lambda) = \sum_j (\mathbf{RP}^n, j\lambda)$$

with each $j \leq n$. As an application we include the following

Proposition. *Given a closed manifold with differentiable involution (M, T) with fixed point set any disjoint union $F = \bigcup \mathbf{RP}^{2n+1}$ (n fixed), then (M, T) bounds.*

Capobianco [1] has shown this is true when $F = \mathbf{RP}^{2m+1}$. Royster [4] proved it when $F = \mathbf{RP}^{2n+1} \cup \mathbf{RP}^{2m+1}$. No such result is possible for $F = \bigcup \mathbf{RP}^{2n}$. Indeed, Stong [5] characterized all involutions fixing $\mathbf{RP}^{2n}$. The author wishes

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PROOF OF THE THEOREM

Let $[\mathbf{RP}^n] \in H_n(\mathbf{RP}^n; \mathbb{Z}_2)$ denote the fundamental class. Let $\tau \rightarrow \mathbf{RP}^n$ denote the tangent bundle. The bordism class $(\mathbf{RP}^n, \xi)$ is completely determined by the characteristic numbers

$$\langle w_I(\xi)w_J(\tau); [\mathbf{RP}^n] \rangle \in \mathbb{Z}_2,$$

where the union $I \cup J$ is a partition of n .

We show first that the classes $(\mathbf{RP}^n, j\lambda)$ of the theorem are linearly independent. Let $\alpha \in H^1(\mathbf{RP}^n; \mathbb{Z}_2)$ denote the nonzero class. For $n = 2m$ consider the $(n+1) \times (n+1)$ matrix whose $(i+1, j+1)$ st entry is the characteristic number

$$\langle w_i(j\lambda)w_1(\tau)^{n-i}; [\mathbf{RP}^n] \rangle = \left\langle \binom{j}{i} \alpha^i \alpha^{n-i}; [\mathbf{RP}^n] \right\rangle = \binom{j}{i} \pmod{2}.$$

This matrix is upper triangular with each diagonal entry equal to 1, so its columns are linearly independent. Thus the classes $\{(\mathbf{RP}^n, j\lambda) | 0 \leq j \leq n\}$ are linearly independent.

For $n = 2m+1$, one must show that the classes $\{(\mathbf{RP}^n, (2j+1)\lambda) | 0 \leq j \leq m\}$ are linearly independent. One argues as above, but uses the $(m+1) \times (m+1)$ matrix with $(i+1, j+1)$ st entry

$$\langle w_{2i+1}((2j+1)\lambda)w_1((2j+1)\lambda)^{n-(2i+1)}; [\mathbf{RP}^n] \rangle.$$

Note that for n odd, the classes $(\mathbf{RP}^n, k\lambda)$ with k even are boundaries, as all nonzero classes $w_I(k\lambda)w_J(\tau)$ are even powers of α , so that all characteristic numbers vanish.

We now must show that for any n , the classes $\{(\mathbf{RP}^n, j\lambda) | j \leq n\}$ span the $\mathbf{RP}^n$ classes in $\mathfrak{N}_n(BO)$. The KO-theory of real bundles tells us that any class $(\mathbf{RP}^n, \xi)$ is bordant to $(\mathbf{RP}^n, k\lambda)$ for some (possibly large) k . If $n = 0$ the theorem is trivial. If $2^a \leq n < 2^{a+1}$, we may assume that $k < 2^{a+1}$ since the total Stiefel-Whitney class $w(2^{a+1}\lambda) = (1 + \alpha)^{2^{a+1}} = 1$. We will show that any class $(\mathbf{RP}^n, k\lambda)$ with $2^a \leq n < k < 2^{a+1}$ is bordant to a disjoint union $\sum_j (\mathbf{RP}^n, j\lambda)$ with $j \leq n$. We first exhibit a decomposition $w(k\lambda) = \sum_j w(j\lambda)$ with each $j \leq n$.

Lemma 1. *Let $2^a \leq n < k < 2^{a+1}$. In the ring $\mathbb{Z}_2[\alpha]/(\alpha^{n+1}) = H^*(\mathbf{RP}^n; \mathbb{Z}_2)$, one has a unique decomposition*

$$(1 + \alpha)^k = (1 + \alpha)^{k-2^a} + \sum_i [(1 + \alpha)^{j_i} + (1 + \alpha)^{2^a+j_i}]$$

with $0 \leq j_i \leq n - 2^a$.

Proof. A $\mathbb{Z}_2$ basis for $\mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ is $\{1, \alpha, \dots, \alpha^n\}$. Since an upper triangular matrix with diagonal entries equal to 1 maps $\alpha^i \mapsto (1 + \alpha)^i$, $0 \leq i \leq n$, it follows that $\{1, (1 + \alpha), \dots, (1 + \alpha)^n\}$ is also a basis. Now

$$(1 + \alpha)^k = (1 + \alpha)^{k-2^a} + \alpha^{2^a}(1 + \alpha)^{k-2^a}.$$

Notice that in the rightmost summand each power of α in $(1 + \alpha)^{k-2^a}$ with exponent greater than $n - 2^a$ is killed by α^{2^a} . By the preceding comment, there is a unique decomposition

$$(1) \quad (1 + \alpha)^{k-2^a} = \sum_t (1 + \alpha)^{j_t}$$

in the ring $\mathbb{Z}_2[\alpha]/(\alpha^{n-2^a+1})$, with $0 \leq j_t \leq n - 2^a$ for each t . Thus

$$\begin{aligned} (1 + \alpha)^k &= (1 + \alpha)^{k-2^a} + \alpha^{2^a} \left[\sum_t (1 + \alpha)^{j_t} \right] \\ &= (1 + \alpha)^{k-2^a} + (1 + 1 + \alpha^{2^a}) \left[\sum_t (1 + \alpha)^{j_t} \right] \\ &= (1 + \alpha)^{k-2^a} + \left[\sum_t (1 + \alpha)^{j_t} + (1 + \alpha)^{2^a+j_t} \right]. \quad \square \end{aligned}$$

There is a useful case where the decomposition has exactly three summands. Suppose $n > 2^a$, so there is a $0 \leq b < a$ with $2^b \leq n - 2^a < 2^{b+1}$. Let $0 \leq k' < 2^{b+1}$ satisfy $k' \equiv k \equiv k - 2^a \pmod{2^{b+1}}$, so that

$$(1 + \alpha)^k = (1 + \alpha)^{k-2^a} + \alpha^{2^a}(1 + \alpha)^{k'}.$$

If $k' \leq n - 2^a$ then

$$(2) \quad (1 + \alpha)^k = (1 + \alpha)^{k-2^a} + [(1 + \alpha)^{k'} + (1 + \alpha)^{2^a+k'}].$$

The following lemma will complete the proof of the theorem:

Lemma 2. *Let $2^a \leq n < k < 2^{a+1}$. The class $(\mathbb{R}P^n, k\lambda) \in \mathfrak{N}_n(BO)$ may be uniquely expressed as a disjoint union*

$$(\mathbb{R}P^n, k\lambda) = (\mathbb{R}P^n, (k - 2^a)\lambda) + \sum_t [(\mathbb{R}P^n, j_t\lambda) + (\mathbb{R}P^n, (2^a + j_t)\lambda)]$$

with the j_t as in Lemma 1.

Proof. Uniqueness follows from the linear independence of the classes on the right-hand side. To show for each partition I and J with union $I \cup J$ a partition of n that

$$\begin{aligned} \langle w_I(k\lambda)w_J(\tau); [\mathbb{R}P^n] \rangle &= \langle w_I((k - 2^a)\lambda)w_J(\tau); [\mathbb{R}P^n] \rangle \\ &\quad + \sum_t [\langle w_I(j_t\lambda)w_J(\tau); [\mathbb{R}P^n] \rangle \\ &\quad + \langle w_I((2^a + j_t)\lambda)w_J(\tau); [\mathbb{R}P^n] \rangle], \end{aligned}$$

it suffices to show that for each partition $I = i_1, i_2, \dots, i_r$ with $|I| = \sum i \leq n$ one has

$$(3) \quad w_I(k\lambda) = w_I((k - 2^a)\lambda) + \sum_t [w_I(j_t\lambda) + w_I((2^a + j_t)\lambda)].$$

Notice that Lemma 1 tells us that the total Stiefel-Whitney classes satisfy the equation

$$w(k\lambda) = w((k - 2^a)\lambda) + \sum_t [w(j_t\lambda) + w((2^a + j_t)\lambda)].$$

In particular, if $I = i_1$ then the result holds.

Case 1. Each $i_s < 2^a$. In this case we have $w_{i_s}(k\lambda) = w_{i_s}((k - 2^a)\lambda)$ and, for each t , $w_{i_s}(j_t\lambda) = w_{i_s}((2^a + j_t)\lambda)$. Hence the product $w_I(k\lambda) = w_I((k - 2^a)\lambda)$ and, for each t , $w_I(j_t\lambda) = w_I((2^a + j_t)\lambda)$, and the lemma holds since we are working over $\mathbb{Z}_2$.

Case 2. $i_1 \geq 2^a$. Note that no more than one of the i_s may be greater than or equal to 2^a since $\sum i \leq n < 2^{a+1}$. We assume without loss of generality that $i_1 \geq 2^a$. Note also that since each $j_t < 2^a$ and $k - 2^a < 2^a$, we have $w_I(j_t\lambda) = w_I((k - 2^a)\lambda) = 0$. So we must show

$$(4) \quad w_I(k\lambda) = \sum_t w_I((2^a + j_t)\lambda).$$

If $n = 2^a$, then $2^a \leq i_1 \leq |I| \leq n = 2^a$, so that $I = i_1 = 2^a$, and the result holds. In particular, if $a = 1$, then $n = 2 = 2^a$ and $k = 3$, and the lemma is true. We proceed with the assumption that $n > 2^a$ and that I has length at least 2 and use induction on a .

Let $0 \leq b < a$ satisfy $2^b \leq n - 2^a < 2^{b+1}$, and let $0 \leq k' < 2^{b+1}$ satisfy $k' \equiv k \equiv k - 2^a \pmod{2^{b+1}}$. Let $K = i_1 - 2^a, i_2, \dots, i_r$. Observe that $|K| \leq n - 2^a$ so that

$$w_{i_s}(k\lambda) = w_{i_s}(k'\lambda) = w_{i_s}((2^a + k')\lambda)$$

for $2 \leq s \leq r$ and

$$w_{i_1}(k\lambda) = \alpha^{2^a} w_{i_1-2^a}(k'\lambda) = \alpha^{2^a} w_{i_1-2^a}((2^a + k')\lambda).$$

If $k' \leq n - 2^a$, then we have

$$w_I(k\lambda) = \alpha^{2^a} w_K(k'\lambda) = w_I((2^a + k')\lambda).$$

By (2) this last equation is equation (4), and we are done.

If $k' > n - 2^a$, write the total class $w(k'\lambda) = w((k - 2^a)\lambda) = \sum_t w(j_t\lambda)$ as in (1). Then

$$w_{i_1}((2^a + j_t)\lambda) = \alpha^{2^a} w_{i_1-2^a}(j_t\lambda)$$

for each t and

$$w_{i_s}((2^a + j_t)\lambda) = w_{i_s}(j_t\lambda)$$

for $2 \leq s \leq r$. Now invoking the inductive hypothesis with $2^b \leq n - 2^a < k' < 2^{b+1}$, we may assume that (3) holds for K , that is, $w_K(k'\lambda) = \sum_t w_K(j_t\lambda)$. But then

$$\begin{aligned} w_I(k\lambda) &= \alpha^{2^a} w_K(k'\lambda) = \alpha^{2^a} \sum_t w_K(j_t\lambda) \\ &= \sum_t \alpha^{2^a} w_K(j_t\lambda) = \sum_t w_I((2^a + j_t)\lambda), \end{aligned}$$

and equation (4) holds. $\square$

PROOF OF THE PROPOSITION

Suppose we are given a closed manifold with differentiable involution (M^m, T) with fixed point set $F = \bigcup \mathbb{R}P^{2n-1}$, where $n \geq 1$ is fixed. We wish to show that the involution (M, T) bounds. Note that if $m = 2n - 1$ then the proposition is trivial, as F consists of connected components of M

each of which bounds in its own right, and T acts freely on the components of $M - F$. So assume that $m > 2n - 1$. In [2] Conner and Floyd showed that the fixed point structure of a differentiable involution on a closed manifold determines the bordism class. That is, if $\mathfrak{N}_m^{\mathbb{Z}_2}$ denotes the group of unoriented bordism classes of involutions on m -manifolds, then the composition

$$\mathfrak{N}_m^{\mathbb{Z}_2} \rightarrow \sum_{k=0}^m \mathfrak{N}_k(\mathbf{BO}(m-k)) \rightarrow \sum_{k=0}^m \mathfrak{N}_k(\mathbf{BO})$$

is monic, where the first map assigns to each involution its fixed point data and the second is induced by the inclusion $\mathbf{BO}(m-k) \hookrightarrow \mathbf{BO}$. Let $\nu \rightarrow F$ denote the normal bundle of F in M . Then either $(F, \nu) = 0 \in \mathfrak{N}_{2n-1}(\mathbf{BO})$ and (M, T) bounds, or according to the theorem we may write

$$(F, \nu) = \sum_i (\mathbf{RP}^{2n-1}, \nu_i) = \sum_i (\mathbf{RP}^{2n-1}, (2k_i - 1)\lambda)$$

with $1 \leq k_i \leq n$ for each i and $k_i \neq k_j$ for $i \neq j$. We will show that the latter is impossible.

So suppose $(F, \nu) = \sum_i (\mathbf{RP}^{2n-1}, (2k_i - 1)\lambda)$ as above. Then note that each $2k_i - 1$ is no greater than the codimension $m - (2n - 1)$ for

$$w_{2k_i-1}(\nu_i^{m-(2n-1)}) = w_{2k_i-1}((2k_i - 1)\lambda) = \alpha^{2k_i-1} \neq 0.$$

According to Kosniowski and Stong [3], if $f(x_1, x_2, \dots, x_m)$ is any symmetric polynomial over $\mathbb{Z}_2$ in m variables of degree at most m , then

$$\begin{aligned} & f(x_1, x_2, \dots, x_m)[M^m] \\ &= \sum_i \frac{f(1+y_1, 1+y_2, \dots, 1+y_{m-(2n-1)}, z_1, z_2, \dots, z_{2n-1})}{(1+y_1)(1+y_2)\cdots(1+y_{m-(2n-1)})} [\mathbf{RP}^{2n-1}] \end{aligned}$$

where i is as above, indexing copies of $\mathbf{RP}^{2n-1}$ in F , and where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_j(x)$, $\sigma_j(y)$, and $\sigma_j(z)$ by the Stiefel-Whitney classes $w_j(M)$, $w_j((2k_i - 1)\lambda)$, and $w_j(\mathbf{RP}^{2n-1})$, respectively and evaluating the resulting cohomology class on the fundamental homology class of M^m or $\mathbf{RP}^{2n-1}$. In our case, since each $2k_i - 1 \leq m - (2n - 1)$, the expression $(1+y_1)(1+y_2)\cdots(1+y_{m-(2n-1)})$ in each denominator on the right is equal to the total class $w((2k_i - 1)\lambda) = (1+\alpha)^{2k_i-1}$.

Let $f(x_1, \dots, x_m) = x_1 + x_2 + \cdots + x_m$. Then for each i

$$\begin{aligned} f(1+y_1, \dots, z_{2n-1}) &= (m - (2n - 1)) + w_1((2k_i - 1)\lambda) + w_1(\mathbf{RP}^{2n-1}) \\ &= (m + 1) + \binom{2k_i - 1}{1} \alpha + \binom{2n}{1} \alpha \\ &= (m + 1) + \alpha. \end{aligned}$$

Now for $1 \leq j \leq n$ let

$$f_j(x_1, \dots, x_m) = (f(x_1, \dots, x_m) + (m + 1))^{2j-1}.$$

Then for each j we have $f_j(x_1, \dots, x_m)[M] = 0$ since no terms in f_j have degree greater than $2j - 1 \leq 2n - 1 < m$. On the other hand, for each i ,

$f_j(1 + y_1, \dots, z_{2n-1}) = ((m+1) + (m+1) + \alpha)^{2j-1} = \alpha^{2j-1}$. So for each $1 \leq j \leq n$ we have

$$0 = \sum_i \frac{\alpha^{2j-1}}{(1+\alpha)^{2k_i-1}} [\mathbf{RP}^{2n-1}].$$

To evaluate this sum, note that for any $k > 0$,

$$\frac{1}{(1+\alpha)^k} = \sum_{t=0}^{2n-1} \binom{k-1+t}{t} \alpha^t$$

in the ring $H^*(\mathbf{RP}^{2n-1}; \mathbf{Z}_2)$, as is seen by induction on k . Hence we have for each j

$$\begin{aligned} 0 &= \sum_i \frac{\alpha^{2j-1}}{(1+\alpha)^{2k_i-1}} [\mathbf{RP}^{2n-1}] \\ (5) \quad &= \sum_i \binom{(2k_i-1)-1+(2n-1)-(2j-1)}{(2n-1)-(2j-1)} \\ &= \sum_i \binom{k_i-1+(n-j)}{(n-j)} \pmod{2}. \end{aligned}$$

But this is impossible. Let A_n be the $n \times n$ matrix over $\mathbf{Z}_2$ with (j, k) th entry $\binom{k-1+(n-j)}{(n-j)}$. Then

$$A_n = \begin{pmatrix} \binom{n-1}{n-1} & \cdots & \binom{2n-2}{n-1} \\ \vdots & & \\ \binom{2}{2} & & \vdots \\ \binom{1}{1} & \binom{2}{1} & & \\ \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{n-1}{0} \end{pmatrix}$$

and A_n is the “upper corner” of Pascal’s triangle reduced modulo 2. If $n = 2^a$, then A_n is lower triangular with each diagonal entry equal to 1, so A_n is nonsingular. If $2^a < n < 2^{a+1}$, then after a finite sequence of elementary row and column operations A_n may be reduced to

$$\begin{pmatrix} 0 & A_{n-2^a} \\ A_{2^a} & 0 \end{pmatrix}$$

and so is nonsingular by induction. In particular, the columns of A_n are linearly independent, so that the sum (5) cannot be 0 for every j with $1 \leq j \leq n$. This is a contradiction. We conclude that $(F, \nu) = 0 \in \mathfrak{N}_{2n-1}(\mathbf{BO})$ so that (M, T) bounds.

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