A COUNTEREXAMPLE TO A VITALI TYPE THEOREM WITH RESPECT TO HAUSDORFF CONTENT

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ABSTRACT. Mateu and Orobitg proved (in *Lipschitz approximation by harmonic functions and some applications to spectral synthesis*, Indiana Univ. Math. J. **39** (1990)) that given $\lambda > 1$ and $d - 1 < \alpha \leq d$ there exist constants C and N (depending on λ and α) with the following property:

For any compact set K in $\mathbb{R}^d$ one can find a (finite) family of balls $\{B(x_i, r_i)\}$ such that (i) $K \subset \bigcup B(x_i, r_i)$, (ii) $\sum r_i^\alpha \leq CM^\alpha(K)$, M^α denoting the α -dimensional Hausdorff content, and (iii) the dilated balls $\{B(x_i, \lambda r_i)\}$ are an almost disjoint family with constant N .

In this paper we prove that such a result is false for $\alpha \leq d - 1$.

1. STATEMENT OF THE RESULT

Given a subset E of $\mathbb{R}^d$ one defines the α -dimensional Hausdorff content of E by

$$M^\alpha(E) = \inf \sum_j l(Q_j)^\alpha,$$

where the infimum is taken over all countable coverings of E by cubes Q_j with sides of length $l(Q_j)$ and parallel to the coordinate axes.

In [MO] the Lip(s) approximation problem by harmonic functions, $0 < s < 1$, was solved using the following [MO, Lemma 2.3].

Covering lemma. *Given $\lambda > 1$ and $d - 1 < \alpha \leq d$ there exist constants C and N (depending on λ and α) such that for any compact set K in $\mathbb{R}^d$ one can find a finite family of cubes $\{Q_i\}$ satisfying:*

- (i) $K \subset \bigcup Q_i$,
- (ii) $\sum_i l(Q_i)^\alpha \leq CM^\alpha(K)$,
- (iii) $\{\lambda Q_i\}$ is an almost disjoint family with constant N .

Remark. If $M^\alpha(K) = 0$ then (ii) should be replaced by $\sum_i l(Q_i)^\alpha \leq \varepsilon$, where $\varepsilon > 0$ has been fixed in advance.

It is said that a family $\{E_j\}$ of subsets of $\mathbb{R}^d$ is almost disjoint with constant N whenever each $x \in \mathbb{R}^d$ belongs to at most N sets E_j . If Q is a cube then λQ denotes the cube of the same center and side length $\lambda l(Q)$.

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In this paper we prove that the above result is false for $0 < \alpha \leq d - 1$. More precisely

Theorem. *Given constants $\lambda > 1$, $C > 1$, $0 < \alpha \leq d - 1$, and a positive integer N there exists a compact set $K = K(\lambda, C, \alpha, N)$ with $M^\alpha(K) > 0$ such that no family of cubes $\{Q_i\}$ can be found satisfying simultaneously:*

- (i) $K \subset \bigcup Q_i$,
- (ii) $\sum_i l(Q_i)^\alpha \leq CM^\alpha(K)$,
- (iii) $\{\lambda Q_i\}$ is an almost disjoint family with constant N .

This proposition answers in the negative a question raised in [MV] in connection with BMO approximation by harmonic functions in the plane. A positive answer to the case $\alpha = d - 1$ would have given a suitable extension of their results to higher dimensions. Thus, new ideas seem to be required to obtain BMO approximation results by harmonic functions in $\mathbb{R}^d$, $d \geq 3$.

In order to free our exposition of notational complications we will only present the proof of the theorem for $d = 2$ and $\alpha = 1$. The general case follows from the same basic ideas with simple technical modifications.

Before coming to the proof of the theorem, some reductions are in order. First let us check that it is enough to prove the theorem only for a fixed $\lambda > 1$. Assume that one has a compact set K and a family of squares $\{Q_i\}$ satisfying (i), (ii), and (iii) for some $\lambda_1 > 1$, $C > 1$, and N . Let $\lambda_2 > \lambda_1$ and let s be the integer part of $1 + (\lambda_2 - 1)/(\lambda_1 - 1)$. We split each square Q_i into s^2 squares of side length $s^{-1}l(Q_i)$, $Q_i = \bigcup_j P_i^j$. Then the family $\{P_i^j\}$ satisfies (i), (ii), and (iii) with $\{Q_i\}$ replaced by $\{P_i^j\}$ and λ_1 , C , and N replaced by λ_2 , sC , and s^2N , respectively. So, one can assume $\lambda = 3$.

Likewise, it will be more convenient to work with a special type of squares instead of all squares with sides parallel to the coordinate axes. We will consider tryadic squares. A tryadic square has a representation $[j3^n, (j+1)3^n] \times [k3^n, (k+1)3^n]$, where j , n , and k are integers. We remark that if Q and P are tryadic squares then either they have disjoint interiors or one contains the other. This fact will be used without further comment in our discussion.

The tryadic 1-dimensional Hausdorff content of a set E of $\mathbb{R}^2$ is defined by

$$M_t^1(E) = \inf \sum_j l(Q_j),$$

where now the infimum is taken over all countable coverings of E by tryadic squares Q_j . Obviously, $M^1(E) \leq M_t^1(E)$ for all sets E . Given a square Q , let Q_i be the tryadic squares that intersect Q with side length 3^j , $3^j \leq l(Q)/2 < 3^{j+1}$, then

- (a) $Q \subset \bigcup Q_i$,
- (b) $\bigcup 3Q_i \subset 3Q$.

Condition (a) gives $M_t^1(E) \leq 9M^1(E)$. From condition (b) it follows that if we have a family of squares covering a set E such that their triples are almost disjoint with constant N , then we can find a family of tryadic squares (of comparable size) covering the same set such that their triples are almost disjoint with constant $9N$.

By the above remarks the only task left is to prove the next

Proposition. *Given a constant $C > 1$ and a positive integer N , there exists a compact set $K = K(C, N)$ with $M_t^1(K) > 0$ such that no family of tryadic squares $\{Q_i\}$ can be found satisfying simultaneously:*

- (i) $K \subset \bigcup Q_i$,
- (ii) $\sum_i l(Q_i) \leq CM_t^1(K)$,
- (iii) $\{3Q_i\}$ is an almost disjoint family with constant N .

2. PROOF OF THE PROPOSITION

Having fixed $C > 1$ and a positive integer N , we will construct a compact subset K of the unit square $I = [0, 1] \times [0, 1]$ such that it will be impossible to find a family of tryadic squares satisfying (i), (ii), and (iii) simultaneously. Our procedure is based on the next lemma.

Main lemma. *Given a tryadic square Q and a constant $C > 1$, let $j = [(C - 1)/2] + 1$ and let m be the first integer such that $3^m \geq 3C$. We put $\alpha = \alpha(C, l(Q)) = l(Q)3^{-j-m}$. Then there exists a positive integer $N(\alpha) \geq 3Cl(Q)/\alpha$ such that for each family of tryadic squares $\{Q_i\}$ satisfying*

- (1) $Q \subset \bigcup Q_i$,
- (2) $\sum_i l(Q_i) \leq Cl(Q)$,
- (3) $l(Q_i) \geq 3\alpha$ if $Q_i \cap Q^c \neq \emptyset$,

one can find $N(\alpha)$ tryadic squares of side length α contained in $(\bigcup 3Q_i) \setminus (\bigcup Q_i)$.

Assume that the main lemma has been proved and let us proceed to the proof of the proposition. Without loss of generality we can suppose the constant C is integer and multiple of 3.

Set $C_1 = 3C$, $\delta_1 = (3C_1)^{-1}$, and for $i = 1, \dots, N$ write

$$\alpha_i = \alpha(C_i, \delta_i) = \delta_i 3^{-j_i - m_i}, \quad \delta_{i+1} = \frac{\delta_i}{3} \alpha_i^2, \quad C_{i+1} = \frac{5}{4} \frac{1}{\delta_i \alpha_i}.$$

Our compact set K will be union of compact sets $K_1, K_2, \dots, K_{N+1}$.

K_1 is the tryadic square in the center of the unit square I with side length δ_1 (see Figure 1).

FIGURE 1

Inside of each tryadic square of side length α_1 in $I \setminus K_1$, consider the n_2 tryadic squares of side length δ_2 with the same center and let K_2 be their union. Thus $M_t^1(K_2) \leq n_2 \delta_2 < \alpha_1^{-2} \delta_2 = \delta_1/3$.

If $i \in \{3, \dots, N+1\}$ then K_i is inductively defined as follows. K_i is the union of all tryadic squares of side length δ_i concentric with the tryadic squares of length α_{i-1} in $I \setminus \bigcup_{j=1}^{i-1} K_j$. Let n_i be the number of them. Thus, $M_t^1(K_i) \leq n_i \delta_i < \alpha_{i-1}^{-2} \delta_i = \delta_{i-1}/3$.

Define

$$K = K_1 \cup K_2 \cup \dots \cup K_{N+1},$$

so that $M_t^1(K) \leq \delta_1 + \delta_1/3 + \dots + \delta_N/3 < \frac{3}{2} \delta_1$.

We are now going to check that K satisfies the desired properties.

Let $\mathcal{Q} = \{Q_i\}$ be some fixed family of tryadic squares such that

- (i) $K \subset \bigcup Q_i$,
- (ii) $\sum l(Q_i) \leq C M_t^1(K) < C_1 \delta_1 \leq 1/2$.

We will show that there exists a point that belongs at least to $N+1$ sets of the family $\{3Q_i\}$. In other words, $\{3Q_i\}$ is not an almost disjoint family with constant N .

Define $\mathcal{F} = \{Q_i \in \mathcal{Q} : Q_i \cap \text{int } K_1 \neq \emptyset \text{ or } l(Q_i) \geq 3\alpha_1\}$. Applying the lemma, we get $N(\alpha_1)$ tryadic squares P with $l(P) = \alpha_1$ contained in $\bigcup_{\mathcal{F}} 3Q_i \setminus \bigcup_{\mathcal{F}} Q_i$. For each of those squares P , one has two possibilities:

- (I) $\sum_{Q_i \subset P} l(Q_i) \leq C_2 \delta_2 < \alpha_1/2$, or
- (II) $\sum_{Q_i \subset P} l(Q_i) > C_2 \delta_2$.

If, for all squares P , (II) holds, then

$$\sum_i l(Q_i) \geq N(\alpha_1) C_2 \delta_2 \geq \frac{3C_1 \delta_1}{\alpha_1} C_2 \delta_2 > C_1 \delta_1,$$

where the second inequality follows from the lemma and the last from the definition of δ_2 and C_2 . Clearly the above inequality contradicts property (ii) $\mathcal{Q}$. Therefore, there exists a tryadic square P_1 with side length α_1 contained in $\bigcup_{\mathcal{F}} 3Q_i \setminus \bigcup_{\mathcal{F}} Q_i$ and satisfying (I). Every point of $P_1 \cap K$ belongs at least to 2 squares in the family $\{3Q_i\}$.

Let $\mathcal{Q}^1 = \{Q_i \in \mathcal{Q} : Q_i \subset P_1\}$. Since

- (i) $K \cap P_1 \subset \bigcup_{\mathcal{Q}^1} Q_i$,
- (ii) $\sum_{\mathcal{Q}^1} l(Q_i) \leq C_2 \delta_2 < l(P_1)/2$,

we can now repeat the argument with $\mathcal{Q}$ replaced by $\mathcal{Q}^1$ and K replaced by $K \cap P_1$.

Repeating the above process, it becomes clear that we can find a point that belongs at least to $N+1$ sets of family $\{3Q_i\}$. This completes the proof of the proposition.

3. PROOF OF THE MAIN LEMMA

We start with two simple remarks.

Remark 1. If one has $Q = \bigcup_i Q_i$, where Q and Q_i are tryadic squares, and one wants to study the set $\bigcup 3Q_i \setminus Q$, then one should only consider squares Q_i that intersect the boundary ∂Q of Q . In other words, if Q_i does not touch ∂Q then $3Q_i \subset Q$.

Remark 2. Keeping the notation in Remark 1, let $\mathcal{B} = \{Q_i: Q_i \cap \partial Q \neq \emptyset\}$. It is a simple calculation to check that if $\sum l(Q_i) \geq 1$, where the sum is taken over all squares $Q_i \in \mathcal{B}$ such that $l(Q_i) \leq l(Q)3^{-k}$, then $\sum_i l(Q_i) \geq (2k+1)l(Q)$.

Let $\mathcal{Q} = \{Q_i\}$ be a family of squares satisfying the conditions (1), (2), and (3) of the lemma. Without loss of generality we can assume that these squares have disjoint interiors. Suppose that there does not exist any $Q_i \in \mathcal{Q}$ that contains Q and subdivide $\mathcal{Q}$ into two families $\mathcal{Q}^1 = \{Q_i \in \mathcal{Q}: Q_i \subset Q\}$ and $\mathcal{Q}^2 = \mathcal{Q} \setminus \mathcal{Q}^1$. We have

$$(1) \quad \sum_{Q_i \in \mathcal{Q}^1} l(Q_i) = Al(Q),$$

where A is a constant less than C . Consider $\mathcal{B} = \{Q_i \in \mathcal{Q}: Q_i \cap \partial Q \neq \emptyset\}$ and define $l = [(A-1)/2] + 1$. Then

$$(2) \quad \sum_{Q_i \in \mathcal{B}, l(Q_i) \leq l(Q)3^{-l}} l(Q_i) < l(Q),$$

because otherwise from Remark 2 we would get

$$\sum_{Q_i \in \mathcal{B}} l(Q_i) \geq (2l+1)l(Q) > Al(Q),$$

which contradicts (1).

We can conclude that the set (see Figure 2)

$$(3) \quad \bigcup_{\mathcal{Q}^1} 3Q_i \setminus \bigcup_{\mathcal{Q}^1} Q_i$$

contains at least $2 \cdot 3^{l-1}$ squares P with $l(P) = l(Q)3^{1-l}$ (touching ∂Q) and

FIGURE 2

FIGURE 3

the union of them has tryadic 1-dimensional Hausdorff content greater than $2l(Q)$, because from (2) it follows that

$$\sum_{Q_i \in \mathcal{B}, l(Q_i) \geq l(Q)3^{-l}} l(Q_i) \geq \frac{5}{3}l(Q).$$

Denote this family of squares by $\mathcal{P}$.

Note that if for some $Q^* \in \mathcal{Q}$ we had $Q \subset Q^*$, then setting $\mathcal{Q}^1 = \{Q^*\}$ we would immediately reach the same conclusion.

We are interested in the set $\bigcup_{\mathcal{Q}} 3Q_i \setminus \bigcup_{\mathcal{Q}} Q_i$, and so far we have only considered (3). We must now look at the squares from the family $\mathcal{Q}^2$. Given $Q_i \in \mathcal{Q}^2$ we have four cases:

- (I) Q_i and P have disjoint interiors for all $P \in \mathcal{P}$.
- (II) $l(Q_i) \geq l(Q)3^{1-l}$.
- (III) There exists some $P \in \mathcal{P}$ such that $Q_i \subset P$ and ∂Q_i touches the exterior side of P , that is, the side of P that lies further from Q (see Figure 3).
- (IV) There exists some $P \in \mathcal{P}$ such that $Q_i \subset P$ and ∂Q_i does not touch the exterior side of P .

If Q_i satisfies (I) or (II) or (III) then we add it to $\mathcal{Q}^1$ and still the property mentioned after (3) holds, that is, $\bigcup_{\mathcal{Q}^1} 3Q_i \setminus \bigcup_{\mathcal{Q}^1} Q_i$ contains a family $\mathcal{P}$ of squares P with $l(P) = l(Q)3^{1-l}$, the number of them is at least $2 \cdot 3^{l-1}$ and their union has tryadic 1-dimensional Hausdorff content bigger than $2l(Q)$.

Therefore, we can assume that if $Q_i \in \mathcal{Q}^2$ then

- (a) Q_i is contained in some $P \in \mathcal{P}$ and $l(Q_i) \leq l(Q)3^{-l}$;
- (b) Q_i does not touch the exterior side of P .

Moreover, recall that we have

- (c) $l(Q_i) \geq 3\alpha$;
- (d) $\sum_{\mathcal{Q}^2} l(Q_i) \leq (C - A)l(Q)$.

FIGURE 4

Now we are able to prove that the number $N(\alpha)$ of squares of side length α inside

$$\left(\bigcup_{\mathcal{Q}^1} 3Q_i \right) \setminus \left(\bigcup_{\mathcal{Q}^1} Q_i \right) \setminus \left(\bigcup_{\mathcal{Q}^2} Q_i \right)$$

is bigger than $3Cl(Q)\alpha^{-1}$. In fact, we are going to see that these squares are inside

$$(4) \quad \bigcup_{\mathcal{P}} P \setminus \bigcup_{\mathcal{Q}^2} Q_i.$$

Owing to properties of family $\mathcal{P}$ and conditions (a), (b), and (c) of family $\mathcal{Q}^2$, it is sufficient (and obviously necessary) to prove that the area of (4) is not less than $2l(Q)^2 3^{-j}$ (because then $N(\alpha) = \alpha^{-2} \text{Area} = \alpha^{-1} 2l(Q) 3^m > \alpha^{-1} 3Cl(Q)$).

The area of the set (4) will be minimized when the squares Q_i are as big as possible, and this will happen whenever in each square P we take 6 squares of side length $l(Q)3^{-l}$, 6 squares of side length $l(Q)3^{1-l}$, and so on until the squares of size $l(Q)3^{1-s-l}$ (see Figure 4).

Consequently, $\sum l(Q_i) \geq 4sl(Q)$ and by condition (d) we have $s \leq (C - A)/4$. Therefore

$$\text{Area} \geq 2 \cdot 3^{l-1} l(Q) 3^{1-l} l(Q) 3^{1-s-l} \geq 2l(Q)^2 3^{1-l-s} \geq 2l(Q)^2 3^{-j},$$

where the last inequality follows from

$$s + l - 1 \leq \frac{C - A}{4} + \frac{A - 1}{2} = \frac{C + A - 2}{4} \leq \frac{C - 1}{2} \leq j.$$

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REFERENCES

- [MO] J. Mateu and J. Orobitg, *Lipschitz approximation by harmonic functions and some applications to spectral synthesis*, Indiana Univ. Math. J. **39** (1990), 703–736.
- [MV] J. Mateu and J. Verdera, *BMO harmonic approximation in the plane and spectral synthesis for Hardy Sobolev spaces*, Rev. Mat. Iberoamericana **4** (1988), 291–318.

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