ON DUALS OF SMOOTH PLANE CURVES

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ABSTRACT. We prove that for given two distinct smooth curves in the projective plane over an algebraically closed field, their duals in the dual plane coincide if and only if they are conics in characteristic two with the same center.

0. INTRODUCTION

In the theory of plane projective geometry in positive characteristic, we are cognizant of many nonclassical phenomena of tangent lines of plane curves. One of the most surprising phenomena is the following fact proved by Wallace [10]. (His discussion was refined by Kleiman and Thorup [7].) Let $\mathbb{P}^2$ be a projective plane over an algebraically closed field of characteristic $p > 0$ and $\check{\mathbb{P}}^2$ the dual plane of $\mathbb{P}^2$. For any given curve in $\check{\mathbb{P}}^2$, any given positive power q of p , and any given positive integer s , there exists a curve $C \subset \mathbb{P}^2$ such that the dual C^* of C coincides with the assigned curve and the Gauss map $\gamma: C \rightarrow C^*$ has separable degree s and inseparable degree q .

In the above situation, if one desires to choose C as smooth, s must be 1 by Kaji's theorem [6]. Moreover, if the assigned curve is a line, then $q = p = 2$ by Lluís's theorem [9].

In his expository work on recent progress on plane curves in characteristic p [8], Kleiman asks, "Is every curve of degree at least two the dual of a smooth curve whose Gauss map has any given inseparable degree q ?"

When the assigned curve in question is smooth, we can see that the answer is negative by observing the list of smooth curves having smooth duals [4, (6.6), (6.7)]. So a small modification should be necessary for the question, that is, we should assume the assigned curve to be singular.

Our results are motivated by the question and show the answer to be negative, even if the question is made into such a modification.

Theorem 1. *Let C_1 and C_2 be distinct smooth curves in $\mathbb{P}^2$. Then $C_1^* = C_2^*$ in $\check{\mathbb{P}}^2$ if and only if C_1 and C_2 are conics in characteristic two with the same center.*

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Theorem 2. Let d^* be a prime number and C a smooth plane curve. Then $\deg C^* = d^*$ if and only if either

- (1) $d^* = 2$, $p \neq 2$, and C is a conic;
- (2) $d^* = 3$, $p = 2$, and C is a cubic; or
- (3) d^* is of form $2^e + 1$ ($e > 0$), $p = 2$, and C is projectively equivalent to the curve $X^{2^e+1} + Y^{2^e+1} + Z^{2^e+1} = 0$.

Kaji's theorem and the standard Plücker formula (for smooth C) are essential tools for our studies.

Kaji's theorem. Let C be a smooth plane curve. Then the Gauss map $\gamma: C \rightarrow C^*$ is purely inseparable.

Proof. See [6, (4.5)].

Plücker formula. Let C be a smooth plane curve of degree d with $\deg C^* = d^*$. Then we have $qd^* = d(d-1)$, where q is the (inseparable) degree of the Gauss map $\gamma: C \rightarrow C^*$.

Proof. By definition, we have $\gamma^* \mathcal{O}_{C^*}(1) \cong \mathcal{O}_C(d-1)$. Calculate the degree of the invertible sheaf in two ways and use Kaji's theorem. Then we have the formula.

1. SMOOTH PLANE CUBICS IN CHARACTERISTIC TWO

First we study smooth plane cubics in characteristic two. The result of this section is used in the proof of Theorem 1, which will be given in §2.

Proposition 3. Let E be a smooth cubic in $\mathbb{P}^2$ over an algebraically closed field of characteristic two. Then E^* is a smooth cubic and, hence, E^{**} is too. Moreover, the composition of Gauss maps $\gamma^* \circ \gamma: E \rightarrow E^{**} \subset \check{\mathbb{P}}^2 = \mathbb{P}^2$ sends the flexes of E onto those of E^{**} , where γ^* is the Gauss map of E^* .

Proof. Thanks to a theorem of Deuring [2], we can start from a simple defining equation of E , that is, E is projectively equivalent to a curve defined by the equation

$$F = \begin{cases} X^3 + Y^2Z + YZ^2 = 0 & \text{if } j = j(E) = 0, \\ jX^3 + XY^2 + XYZ + Z^3/j = 0 & \text{if } j = j(E) \neq 0, \end{cases}$$

where j is the j -invariant of E .

The first part of our assertion was proved in [4, (6.7)]. (Note that there is a small mistake in that proof; $V = \partial F / \partial Y$ and $W = \partial F / \partial Z$ should read as $V = \partial F / \partial Z$ and $W = \partial F / \partial Y$, respectively.) Putting $U = \partial F / \partial X$, $V = \partial F / \partial Z$, and $W = \partial F / \partial Y$, a defining equation of the dual E^* of E is

$$\begin{cases} U^3 + V^2W + VW^2 = 0 & \text{if } j(E) = 0, \\ UV^2 + UVW + U^2W/j + W^3 = 0 & \text{if } j(E) \neq 0. \end{cases}$$

If $j = 0$, then $\gamma(x, y, z) = (x^2, y^2, z^2)$ for any $(x, y, z) \in E$. Hence $\gamma: E \rightarrow E^*$ can be extended to a purely inseparable morphism $\mathbb{P}^2 \rightarrow \check{\mathbb{P}}^2$. Observing the equation of E^* and using the above fact, $\gamma^*: E^* \rightarrow E^{**}$ is also extendable to a purely inseparable morphism $\check{\mathbb{P}}^2 \rightarrow \check{\mathbb{P}}^2 = \mathbb{P}^2$; therefore, $\gamma^* \circ \gamma: E \rightarrow E^{**}$ can be extended to a Frobenius morphism $\mathbb{P}^2 \rightarrow \mathbb{P}^2$ and hence sends the flexes of E onto those of E^{**} .

Now we consider the case $j(E) \neq 0$. Change coordinates of $\check{\mathbb{P}}^2$ as

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} j^2 & 0 & 0 \\ j & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ V_1 \\ W_1 \end{pmatrix}.$$

Then E^* is defined by the equation

$$G = j^2 U_1^3 + U_1 V_1^2 + U_1 V_1 W_1 + W_1^3 / j^2 = 0.$$

Putting $X' = \partial G / \partial U_1$, $Y' = \partial G / \partial V_1$, and $Z' = \partial G / \partial W_1$ and changing coordinates of $\check{\mathbb{P}}^2 = \mathbb{P}^2$ to

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} j^4 & 0 & 0 \\ j^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{pmatrix},$$

an equation of E^{**} is

$$H = j^4 \bar{X}^3 + \bar{X} \bar{Y}^2 + \bar{X} \bar{Y} \bar{Z} + \bar{Z}^3 / j^4 = 0.$$

Now we compute $\gamma^* \circ \gamma(x, y, z)$ for $(x, y, z) \in E$. We denote coordinates of points which are the image of (x, y, z) by small letters corresponding to capital letters of the frame of reference.

Hence we have

$$\begin{cases} \bar{x} = \frac{1}{j^4} x' = \frac{1}{j^2} u_1^2 + \frac{1}{j^4} v_1^2 + \frac{1}{j^4} v_1 w_1, \\ \bar{y} = \frac{1}{j^2} x' + y' = u_1^2 + \frac{1}{j^2} v_1^2 + \frac{1}{j^2} v_1 w_1 + u_1 v_1 + \frac{1}{j^2} w_1^2, \\ \bar{z} = z' = u_1 w_1 \end{cases}$$

and

$$\begin{cases} u_1 = \frac{1}{j^2} u = \frac{1}{j} x^2 + \frac{1}{j^2} y^2 + \frac{1}{j^2} y z, \\ v_1 = \frac{1}{j} u + v = x^2 + \frac{1}{j} y^2 + \frac{1}{j} y z + x y + \frac{1}{j} z^2, \\ w_1 = w = x z. \end{cases}$$

Using the equation $j x^3 + x y^2 + x y z + z^3 / j = 0$, we have $\bar{x} = x^4 / j^3$, $\bar{y} = y^4 / j^3$, and $\bar{z} = z^4 / j^3$. This means that $\gamma^* \circ \gamma: E \rightarrow E^{**}$ can be extended to a Frobenius morphism $\mathbb{P}^2 \rightarrow \mathbb{P}^2$. This completes the proof.

Remark. The image of each flex of E under the Gauss map $\gamma: E \rightarrow E^* \subset \check{\mathbb{P}}^2$ is not a flex of E^* if $j(E) \neq 0$.

2. PROOF OF THEOREM 1

Now we prove Theorem 1. Since the *if* part of the theorem is trivial, we give a proof of the *only if* part. We denote by d_i the degree of C_i .

First we note that $d_1 = d_2$. In fact, since Gauss maps γ_1 of C_1 and γ_2 of C_2 are purely inseparable by Kaji's theorem, we have that the geometric genus of C_i coincides with that of C_i^* for $i = 1, 2$. Hence we have $d_1 = d_2$ because both plane curves are smooth and have the same genus.

Let q_i be the (inseparable) degree of γ_i for $i = 1, 2$. Since $C_1^* = C_2^*$ and $d_1 = d_2$, we have $q_1 = q_2$ by the Plücker formula. This means that the reflexivity of C_1 coincides with that of C_2 . It is obvious that if both C_1 and C_2 are reflexive, then $C_1^* \neq C_2^*$. So we may assume that C_1 and C_2 are nonreflexive.

Let us consider the correspondence

$$G' = \{(P, Q) \in C_1 \times C_2 \mid T_P C_1 = T_Q C_2\}$$

with reduced structure, where $T_P C$ means the embedded tangent line to a curve C at a point $P \in C$. Obviously, the diagram

$$\begin{array}{ccc} G' & \xrightarrow{p'_2} & C_2 \\ p'_1 \downarrow & & \downarrow \gamma_2 \\ C_1 & \xrightarrow{\gamma_1} & C_1^* = C_2^* \end{array}$$

is commutative, where p'_1 and p'_2 are projections. Note that $p_i: G' \rightarrow C_i$ ($i = 1, 2$) are surjective by our assumption $C_1^* = C_2^*$ and generically one-to-one by Kaji's theorem. Hence $\dim G' = 1$ and a one-dimensional component of G' is unique. We denote by G the one-dimensional component of G' and by p_i the restriction of p'_i to G .

We show that p_1 and p_2 are separable. Let us consider the commutative diagram of Zariski tangent spaces for a general point $(P, Q) \in G$:

$$\begin{array}{ccc} & & \mathcal{T}_P C_1 \\ & \nearrow \text{1st projection} & \uparrow dp_1 \\ \mathcal{T}_{(P,Q)} C_1 \times C_2 = \mathcal{T}_P C_1 \times \mathcal{T}_Q C_2 & \longleftrightarrow & \mathcal{T}_{(P,Q)} G \\ & \searrow \text{2nd projection} & \uparrow dp_2 \\ & & \mathcal{T}_Q C_2 \end{array}$$

Therefore, $dp_1 \neq 0$ or $dp_2 \neq 0$. Hence, p_1 or p_2 is separable. Since $\gamma_1 \circ p_1 = \gamma_2 \circ p_2$ and γ_1 and γ_2 are purely inseparable of the same degree, both p_1 and p_2 are separable; therefore, both p_1 and p_2 are birational. Since C_1 and C_2 are smooth, p_1 and p_2 are isomorphisms. Let $\sigma = p_2 \circ p_1^{-1}: C_1 \xrightarrow{\sim} C_2$. We need the following

Lemma. *There is an automorphism $\tilde{\sigma}$ of $\mathbb{P}^2$ such that the diagram*

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\tilde{\sigma}} & \mathbb{P}^2 \\ \uparrow & & \uparrow \\ C_1 & \xrightarrow{\sigma} & C_2 \end{array}$$

is commutative.

Proof. Since C_1 and C_2 are smooth plane curves, both curves are linearly normal, i.e., $H^0(\mathbb{P}^2, \mathcal{O}(1)) \rightarrow H^0(C_i, \mathcal{O}_{C_i}(1))$ is an isomorphism. So it suffices to show that $\sigma^* \mathcal{O}_{C_2}(1) \cong \mathcal{O}_{C_1}(1)$. When $d \geq 4$, this is guaranteed by [1, p. 56, Exercise 18] and when $d = 2$, this is obvious.

Now we prove the claim when $d = 3$ under our situation. Since C_1 and C_2 are nonreflexive smooth cubic, we have $p = 2$. By Proposition 3, the map $C_i \xrightarrow{\gamma_i} C_i^* \xrightarrow{\gamma_i^*} C_i^{**}$ sends the flexes of C_i onto those of C_i^{**} for $i = 1, 2$. Since

the diagram

$$\begin{array}{ccc}
 C_1 & \xrightarrow{\sigma} & C_2 \\
 \gamma_1 \searrow & & \swarrow \gamma_2 \\
 & C_1^* = C_2^* & \\
 & \downarrow & \\
 & C_1^{**} = C_2^{**} &
 \end{array}$$

is commutative, σ sends the flexes of C_1 onto those of C_2 . Since $\mathcal{O}_{C_i}(1) \cong \mathcal{O}_{C_i}(3R)$ for each flex $R \in C_i$, we have $\sigma^*\mathcal{O}_{C_2}(1) \cong \mathcal{O}_{C_1}(1)$. $\square$

Let us return to the proof of Theorem 1. Let $P \in C_1$. Then $\tilde{\sigma}(T_P C_1) = T_{\sigma(P)} C_2$. Since $T_P C_1 = T_{\sigma(P)} C_2$ by our assumption, we have $\tilde{\sigma}(T_P C_1) = T_P C_1$. Consider the linear transformation of the dual plane $\check{\sigma}^{-1}: \check{\mathbb{P}}^2 \rightarrow \check{\mathbb{P}}^2$. Then we have $\check{\sigma}^{-1}((T_P C_1)^*) = (T_P C_1)^*$ for each $P \in C_1$. Hence we have that $\{(T_P C_1)^* | P \in C_1\}$ are collinear in $\check{\mathbb{P}}^2$ because $\check{\sigma}^{-1}$ is not the identity map. Therefore $C_1^* = C_2^*$ is a line. This means that C_1 and C_2 are conics in characteristic two with the same center by Lluís's theorem [9]. This completes the proof.

3. SMOOTH PLANE CURVES WITH AN ASSIGNED CLASS

The purpose of this section is to prove Theorem 2. Let d^* be a given prime number and C a smooth plane curve of degree d with $\deg C^* = d^*$. By the Plücker formula, we have

$$p^e d^* = d(d-1),$$

where p^e is the (inseparable) degree of the Gauss map of C .

First we consider the case $p \neq 2$. Since $d(d-1)$ is divisible by 2, we have $d^* = 2$ because d^* is prime. On the other hand, since $p^e \leq d$, we have $2d \geq d(d-1)$ by the Plücker formula. Hence we have $d = 2$ or 3. If $d = 2$, then (1) occurs. If $d = 3$, then $p^e = d = 3$, but such a curve does not exist because there are no smooth curves with $p^e = d > 2$ [4, (3.4)].

Next we consider the case $p = 2$. If $e > 1$, then $d = k \cdot 2^e + 1$ by the generalization of Pardini's theorem [3, 5]. Hence we have

$$d^* = k(k \cdot 2^e + 1).$$

Since d^* is prime, we have $k = 1$ and $d^* = 2^e + 1$. In this case, (3) occurs by the latter part of the generalization of Pardini's theorem. If $e = 1$, then we have $2d^* = d(d-1)$. Since d^* is prime, we have

$$2 = d - 1 \text{ and } d^* = d, \quad \text{which is a prime number,}$$

or

$$2 = d \text{ and } d^* = d - 1, \quad \text{which is a prime number.}$$

In the former case, (2) occurs. The latter case does not occur because $d^* = 1$. If $e = 0$, then $d^* = d(d-1)$. Since d^* is prime, we have $d = 2$. So C is a conic in characteristic two, but such a curve is nonreflexive, i.e., $e > 0$. This is a contradiction.

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