THE FAT LOCUS OF HILBERT SCHEMES OF POINTS

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ABSTRACT. Let X be a smooth projective variety over an algebraically closed field k . Let $\text{Hilb}^d(X)$ be its Hilbert scheme of 0-dimensional subschemes of X of degree d . Let $[\text{Hilb}^d(X)](k)$ be the set of k -rational points. We prove that the subset of points of $[\text{Hilb}^d(X)](k)$ corresponding to fat points on X is a locally closed subset with respect to the Zariski topology.

Let X be a smooth d -dimensional projective variety defined over an algebraically closed field k . For $n \in \mathbb{Z}_{\geq 1}$ let $\text{Hilb}^n(X)$ be the Hilbert scheme parametrizing 0-dimensional subschemes Z of X of degree $\deg(Z) = n$ (i.e., $\sum(l(\mathcal{O}_{Z;P}) : P \in X(k)) = n$, where $X(k)$ is the set of the closed points on X and l means length). We also write Z for the corresponding point on $[\text{Hilb}^n(X)](k)$.

Definition. A 0-dimensional closed subscheme Z of X is called fat of type $(n_1; n_2; \dots; n_t)$ if $n_1 \geq n_2 \geq \dots \geq n_t \geq 1$ and there exist t different points $P_1; \dots; P_t$ on $X(k)$ such that

$$\begin{aligned}\mathcal{I}_{Z;P} &= \mathcal{O}_{X;P} \quad \text{for } P \in X(k) \setminus \{P_1; \dots; P_t\}, \\ \mathcal{I}_{Z;P_i} &= (\mathcal{M}_{X;P_i})^{n_i}\end{aligned}$$

where $\mathcal{I}_Z$ is the sheaf of ideals of $Z \subset X$ and $\mathcal{M}_{X;P_i}$ is the maximal ideal in the local ring $\mathcal{O}_{X;P_i}$ of X at P_i .

Since X is smooth, we have $\deg(Z) = \sum((n_i + d - 1) : 1 \leq i \leq t)$ in the above definition. For $n = \sum((n_i + d - 1) : 1 \leq i \leq t)$ we define

$$\text{Hilb}(n_1; \dots; n_t) = \{Z \in (\text{Hilb}^n(X))(k) : Z \text{ is fat of type } (n_1; \dots; n_t)\}.$$

We call $\text{Hilb}(n_1; \dots; n_t)$ the fat locus of type $(n_1; \dots; n_t)$. In this paper, we prove

Theorem. $\text{Hilb}(n_1; \dots; n_t)$ is a locally closed subset of $[\text{Hilb}^n(X)](k)$, i.e., it is the intersection $Z \cap U$ of a closed subset Z and an open subset U of $[\text{Hilb}^n(X)](k)$. (Of course, we work with respect to the Zariski topology.)

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The motivation comes from a growing interest in fat points (see, e.g., [1, 2]). In particular, it is proved in [5] that $\text{Hilb}(n_1; n_2; \dots; n_t)$ is an irreducible constructible subset of $[\text{Hilb}^n(X)](k)$ (it is only stated for the case $X = \mathbb{P}^2$ but his proof works in our generality), and the theorem proved in this paper is also mentioned in [5] as a well-known but not yet proved statement.

Proof. For $m \in \mathbb{Z}_{\geq 1}$ and $n = \binom{m+d-1}{d}$ it is remarked in [5] that $\text{Hilb}(m)$ is a closed subset of $[\text{Hilb}^n(X)](k)$. So we can assume $t \geq 2$ and are going to use induction on t . We abbreviate $(n_1; \dots; n_t)$ by $\mathbf{n}$ and $\text{Hilb}(n_1; \dots; n_t)$ by $\text{Hilb}(\mathbf{n})$.

Lemma 1. *For each irreducible component H of $[\text{Hilb}^n(X)](k)$ it is enough to prove that $H \cap \text{Hilb}(\mathbf{n})$ is locally closed in H .*

Proof. Let $H^1, \dots, H^s$ be the irreducible components of $\text{Hilb}^n(X)$ and let $H^i(\mathbf{n}) = H^i \cap \text{Hilb}(\mathbf{n})$. Since $H^i(\mathbf{n})$ is locally closed in H^i , we have $H^i(\mathbf{n}) = U'_i \cap C_i$, where $U'_i \subset H^i$ is open and $C_i \subset H^i$ is closed. Since H^i is closed in $[\text{Hilb}^n(X)](k)$, one has that $C_i \subset [\text{Hilb}^n(X)](k)$ is closed. Moreover, there exists $U_i \subset [\text{Hilb}^n(X)](k)$ open with $U'_i = U_i \cap H^i$. We can assume $U_i \supset [\text{Hilb}^n(X)](k) \setminus H^i$.

Let $U = U_1 \cap \dots \cap U_s \subset [\text{Hilb}^n(X)](k)$ be open and $C = C_1 \cup \dots \cup C_s \subset [\text{Hilb}^n(X)](k)$ be closed. Take $Z \in \text{Hilb}(\mathbf{n})$ and let $1 \leq i \leq s$. If $Z \notin H^i$ then $Z \in U_i$. If $Z \in H^i$ then $Z \in H^i(\mathbf{n}) \subset U'_i \cap C_i$, so $Z \in U_i$ and $Z \in C_i$. This already proves $Z \in U_1 \cap \dots \cap U_s = U$ and $Z \in C_1 \cup \dots \cup C_s = C$, and we prove $\text{Hilb}(\mathbf{n}) \subset U \cap C$.

Now, let $Z \in U \cap C$. Since $Z \in C = C_1 \cup \dots \cup C_s$, there exists $1 \leq i \leq s$ with $Z \in C_i \subset H^i$. Since $Z \in U = U_1 \cap \dots \cap U_s$, we also have $Z \in U_i$, so $Z \in U_i \cap H^i = U'_i$. This proves $Z \in U'_i \cap C_i = H^i(\mathbf{n})$, hence $Z \in \text{Hilb}(\mathbf{n})$. This completes the proof that $\text{Hilb}(\mathbf{n}) = U \cap C$.

Now we take an irreducible component H of $\text{Hilb}^n(X)$, and we let $H(\mathbf{n}) = H \cap \text{Hilb}(\mathbf{n})$. We are going to prove that $H(\mathbf{n})$ is locally closed in $H(k)$. The restriction to H of the universal family of closed subschemes of X over $\text{Hilb}^n(X)$ gives us

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & H \times X \\ & \searrow p & \\ & & H \end{array}$$

Let $\mathcal{I}_{\mathcal{C}}$ be the sheaf of ideals of $\mathcal{C}$ in $\mathcal{O}_{H \times X}$. Consider the following diagram (pr_{ij} means projection on the i th and j th factors):

$$\begin{array}{ccccc} H \times X & \xleftarrow{\text{pr}_{13}} & H \times X \times X & \xrightarrow{\text{pr}_{23}} & X \times X \\ & & \downarrow \text{pr}_{12} & & \\ & & H \times X & & \end{array}$$

Let $\mathcal{I}_{\Delta}$ be the sheaf of ideals of the diagonal $\Delta \subset X \times X$. Let $\mathcal{I}_{n_1 \Delta}$ be the image of $\mathcal{I}_{\Delta}^{\otimes n_1}$ in $\mathcal{O}_{X \times X}$. Let $\mathcal{I}'_{n_1 \Delta} = \mathcal{I}_{n_1 \Delta} \cdot \mathcal{O}_{H \times X \times X}$ using pr_{23} and let $\mathcal{I}'_{\mathcal{C}} = \mathcal{I}_{\mathcal{C}} \cdot \mathcal{O}_{H \times X \times X}$ using pr_{13} (see [3, p. 163]). Let $\mathcal{I}$ be the sum $\mathcal{I}'_{n_1 \Delta} + \mathcal{I}'_{\mathcal{C}}$

in $\mathcal{O}_{H \times X \times X}$, and let $\mathcal{S} \subset H \times X \times X$ be the associated closed subscheme. Consider the restriction of pr_{12} ,

$$q : \mathcal{S} \rightarrow H \times X.$$

Lemma 2. For $(Z; P) \in H(k) \times X$, the scheme-theoretic inverse image $q^{-1}(Z; P)$ is the closed subscheme of X with sheaf of ideals $\mathcal{I}(Z; P)$ given by

$$\begin{aligned} \mathcal{I}(Z; P)_Q &= \mathcal{O}_{X; Q} \quad \text{for } Q \in X(k) \setminus \{P\}, \\ \mathcal{I}(Z; P)_P &= \mathcal{I}_{Z; P} + \mathfrak{M}_{X; P}^{n_1}. \end{aligned}$$

Proof. For $Q \in X(k) \setminus \{P\}$ one has $(\mathcal{I}_{n_1 \Delta})(P; Q) = \mathcal{O}_{X \times X; (P; Q)}$ so $(\mathcal{I}'_{n_1 \Delta})(Z; P; Q) = \mathcal{O}_{H \times X \times X; (Z; P; Q)}$. Thus $\mathcal{I}(Z; P; Q) = \mathcal{O}_{H \times X \times X; (Z; P; Q)}$ and, therefore, also $\mathcal{I}(Z; P)_Q = \mathcal{O}_{X; Q}$. On the other hand, let $t_1, \dots, t_d$ be a system of local parameters in $\mathcal{O}_{X; P}$. This induces a system of parameters $t_1^{(1)}, \dots, t_d^{(1)}, t_1^{(2)}, \dots, t_d^{(2)}$ in $\mathcal{O}_{X \times X; (P; P)}$ and $(\mathcal{I}_{n_1 \Delta})(P; P) = \langle t_1^{(1)} - t_1^{(2)}, \dots, t_d^{(1)} - t_d^{(2)} \rangle^{n_1}$, so also $(\mathcal{I}'_{n_1 \Delta})(Z; P; P)$ is generated by the images of degree n_1 -monomials in $t_i^{(1)} - t_i^{(2)}$ ($1 \leq i \leq d$) in $\mathcal{O}_{H \times X \times X; (Z; P; P)}$. Moreover, $(\mathcal{I}'_{\mathcal{S}})(Z; P; P)$ can be described as follows. Consider the natural local homomorphism

$$\mathcal{O}_{H \times X; (Z; P)} \xrightarrow{\text{pr}_{13}^\dagger} \mathcal{O}_{H \times X \times X; (Z; P; P)}.$$

Then $(\mathcal{I}'_{\mathcal{S}})(Z; P; P)$ is induced by $(\mathcal{I}'_{\mathcal{S}})(Z; P)$. Let $\mathfrak{M}'_{(Z; P; P)}$ be the ideal induced by $\mathfrak{M}_{H \times X; (Z; P)} < \mathcal{O}_{H \times X; (Z; P)}$ under

$$\mathcal{O}_{H \times X; (Z; P)} \xrightarrow{\text{pr}_{12}^\dagger} \mathcal{O}_{H \times X \times X; (Z; P; P)}.$$

Then $(\mathcal{I}(Z; P))_P = (\mathfrak{M}'_{(Z; P; P)} + (\mathcal{I}'_{\mathcal{S}})(Z; P; P) + (\mathcal{I}'_{n_1 \Delta})(Z; P; P)) / \mathfrak{M}'_{(Z; P; P)}$ in $\mathcal{O}_{H \times X \times X; (Z; P; P)} / \mathfrak{M}'_{(Z; P; P)} = \mathcal{O}_{X; P}$. Let $\mathfrak{M}''(Z)$ be induced by $\mathfrak{M}_{H; Z}$ under $\mathcal{O}_{H; Z} \xrightarrow{\text{pr}_1^\dagger} \mathcal{O}_{H \times X \times X; (Z; P; P)}$. Then $\mathfrak{M}''(Z) + (\mathcal{I}'_{\mathcal{S}})(Z; P; P)$ is obtained from $\mathfrak{M}''(Z) + \mathcal{I}_{\mathcal{S}; (Z; P)}$ under $\mathcal{O}_{H \times X; (Z; P)} \xrightarrow{\text{pr}_{13}^\dagger} \mathcal{O}_{H \times X \times X; (Z; P; P)}$, where $\mathfrak{M}''(Z)$ is obtained from $\mathfrak{M}_{H; Z}$ under $\mathcal{O}_{H; Z} \xrightarrow{\text{pr}_1^\dagger} \mathcal{O}_{H \times X; (Z; P)}$. Since

$$(\mathfrak{M}''(Z) + \mathcal{I}_{\mathcal{S}; (Z; P)}) / \mathfrak{M}''(Z) = \mathcal{I}_{Z; P} < \mathcal{O}_{H \times X; (Z; P)} / \mathfrak{M}''(Z) = \mathcal{O}_{X; P},$$

one has

$$(\mathcal{I}(Z; P))_P = (\mathcal{I}'_{(Z; P)} + \mathfrak{M}'_{(P; P)} + (\mathcal{I}_{n_1 \Delta})(P; P)) / \mathfrak{M}'_{(P; P)},$$

where $\mathcal{I}'_{(Z; P)}$ (resp. $\mathfrak{M}'_{(P; P)}$) is induced from $\mathcal{I}_{Z; P}$ (resp. $\mathfrak{M}_{X; P}$) under $\mathcal{O}_{X; P} \xrightarrow{\text{pr}_2^\dagger} \mathcal{O}_{X \times X; (P; P)}$ (resp. $\mathcal{O}_{X; P} \xrightarrow{\text{pr}_1^\dagger} \mathcal{O}_{X \times X; (P; P)}$). From the description of $(\mathcal{I}_{n_1 \Delta})(P; P)$ we obtain

$$(\mathcal{I}(Z; P))_P = \mathcal{I}_{Z; P} + \langle t_1, \dots, t_d \rangle^{n_1} = \mathcal{I}_{Z; P} + \mathfrak{M}_{X; P}^{n_1}.$$

Remember the concept of a flattening stratification of $\mathcal{I}_{\mathcal{S}}$ over $H \times X$ (under q) (see [4, Lecture 8; 6, §6]). This implies that $H \times X$ is the union of finitely many locally closed subsets where each such subset corresponds to some prescribed degree for the fibre of $\mathcal{S}$. Clearly $H'(n_1) = \{(Z; P) \in H \times X : \deg(\mathcal{I}_{(Z; P)}) = \binom{n_1 + d - 1}{d}\}$ is one such stratum (here $\mathcal{I}_{(Z; P)}$ is the associated closed subscheme of $X \times_{\text{Spec}(k)} \text{Spec}(k(Z; P))$).

Lemma 3. $(Z; P) \in [H'(n_1)](k)$ if and only if $\mathcal{I}_{Z;P} \subset \mathfrak{M}_{X;P}^{n_1}$.

Proof. Since $(\mathcal{I}(Z; P))_P \supset \mathfrak{M}_{X;P}^{n_1}$ and $l(\mathcal{O}_{X;P}/\mathfrak{M}_{X;P}^{n_1}) = \binom{n_1+d-1}{d}$, one has $(Z; P) \in H'(n_1)$, i.e., $l(\mathcal{O}_{X;P}/(\mathcal{I}(Z; P)_P)) = \binom{n_1+d-1}{d}$, iff $(\mathcal{I}(Z; P))_P = \mathfrak{M}_{X;P}^{n_1}$. Because of Lemma 2, this is equivalent to $\mathcal{I}_{Z;P} \subset \mathfrak{M}_{X;P}^{n_1}$.

Lemma 4. $[H'(n_1)](k)$ is a closed subset of $(H \times X)(k)$.

Proof. Using pr_{12} , $\mathcal{O}_{H \times X \times X}/\mathcal{I}'_{n_1\Delta}$ is a coherent $\mathcal{O}_{H \times X}$ -module. But $\mathcal{O}_{\mathcal{I}} = \mathcal{O}_{H \times X \times X}/\mathcal{I}$ is a quotient of it, hence it is also a coherent $\mathcal{O}_{H \times X}$ -module. This implies that $H \times X \rightarrow \mathbb{N}: (Z; P) \rightarrow \deg(\mathcal{I}(Z; P))$ is upper semicontinuous on $H \times X$ (see, e.g., [6, pp. 1–10]). Since $\binom{n_1+d-1}{d}$ is the maximal possible value, we obtain our claim.

Corollary. Since X is a projective variety, the projection $H \times X \xrightarrow{\text{pr}_1} H$ is closed. So $\text{pr}_1([H'(n_1)](k)) \subset H(k)$ is closed. Clearly $H(\mathbb{n}) \subset \text{pr}_1([H'(n_1)](k))$.

We consider now the following situation:

$$\begin{array}{ccccccc}
 Z(\mathcal{I}_{n_1\Delta}) & \longleftarrow & \text{pr}_{23}^{-1}(Z(\mathcal{I}_{n_1\Delta})) & \hookleftarrow & \mathcal{I}' & \hookrightarrow & \mathcal{I} \\
 & & & & \downarrow & & \downarrow \\
 \cap & & \cap & & \mathcal{E}' & \hookrightarrow & \text{pr}_{13}^{-1}(\mathcal{E}) \longrightarrow \mathcal{E} \\
 & & & & \cap & & \cap \\
 X \times X & \xleftarrow{\text{pr}_{23}} & H \times X \times X & \supset & H'(n_1) \times X \subset & H \times X \times X & \xrightarrow{\text{pr}_{13}} H \times X \\
 & & & & \downarrow & & \downarrow \text{pr}_{12} \\
 & & & & H'(n_1) & \subset & H \times X
 \end{array}$$

(here Z means closed subscheme defined by the mentioned sheaf of ideals and $\mathcal{E}'$ (resp. $\mathcal{I}'$) are restrictions—scheme theoretical—of $\text{pr}_{13}^{-1}(\mathcal{E})$ (resp. $\mathcal{I}$) to $H'(n_1) \times X$). Let $\mathcal{I}'_{(n_1+1)\Delta} = \mathcal{I}_{(n_1+1)\Delta} \cdot \mathcal{O}_{H'(n_1) \times X}$ using pr_{23} . Let $\mathcal{I}_1 = \mathcal{I}'_{(n_1+1)\Delta} + \mathcal{I}_{\mathcal{E}'}$ in $\mathcal{O}_{H'(n_1) \times X}$ and let $\mathcal{I}_1 \subset H'(n_1) \times X$ be the associated closed subscheme. The projection on $H'(n_1)$ gives a morphism $q_1: \mathcal{I}_1 \rightarrow H'(n_1)$.

Lemma 2'. For $(Z; P) \in [H'(n_1)](k) \rightarrow q_1^{-1}(Z; P)$ scheme-theoretical and considered as a closed subscheme of X —has a sheaf of ideals $\mathcal{I}_1(Z; P) \subset \mathcal{O}_X$ given by

$$\begin{aligned}
 (\mathcal{I}_1(Z; P))_Q &= \mathcal{O}_{X;Q} \quad \text{for } Q \in X(k) \setminus \{P\}, \\
 (\mathcal{I}_1(Z; P))_P &= \mathcal{I}_{Z;P} + \mathfrak{M}_{X;P}^{n_1+1}.
 \end{aligned}$$

The proof is the same as the proof of Lemma 2.

Since $(\mathcal{I}_1(Z; P))_P \subset (\mathcal{I}(Z; P))_P$, one has $\deg(\mathcal{I}_1(Z; P)) \geq \binom{n_1+d-1}{d}$ for each $(Z; P) \in [H'(n_1)](k)$. Moreover, being a quotient of $\mathcal{O}_{H'(n_1) \times X}/\mathcal{I}'_{(n_1+1)\Delta}$, $\mathcal{O}_{\mathcal{I}_1}$ is coherent as an $\mathcal{O}_{H'(n_1)}$ -module. This implies that

$$H''(n_1) = \left\{ (Z; P) \in H'(n_1) : \deg(\mathcal{I}_1(Z; P)) > \binom{n_1+d-1}{d} \right\} \subset H'(n_1)$$

is closed. So $H''(n_1) \subset H \times X$ is closed and $\text{pr}_1(H''(n_1)) \subset H$ is closed. For $Z \in [\text{pr}_1(H''(n_1))](k)$, there exists $P \in X(k)$ with $l(\mathcal{O}_{Z;P}) > \binom{n_1+d-1}{d}$, so

$Z \notin H(n)$. This implies

$$H(n) \subset [\mathrm{pr}_1(H'(n_1))](k) \setminus [\mathrm{pr}_1(H''(n_1))](k).$$

We write $H(n_1) = H'(n_1) \setminus H''(n_1)$.

Lemma 5. For $(Z; P) \in [H(n_1)](k)$, one has $\mathcal{I}_{Z;P} = \mathfrak{M}_{X;P}^{n_1}$.

Proof. Because of Lemma 3, one has $\mathcal{I}_{Z;P} \subset \mathfrak{M}_{X;P}^{n_1}$. This implies $\mathcal{I}_{Z;P} + \mathfrak{M}_{X;P}^{n_1+1} \subset \mathfrak{M}_{X;P}^{n_1}$. Moreover,

$$l(\mathcal{O}_{X;P}/\mathcal{I}_{Z;P} + \mathfrak{M}_{X;P}^{n_1+1}) = l(\mathcal{O}_{X;P}/\mathfrak{M}_{X;P}^{n_1}),$$

hence $\mathcal{I}_{Z;P} + \mathfrak{M}_{X;P}^{n_1+1} = \mathfrak{M}_{X;P}^{n_1}$. This implies

$$\mathfrak{M}_{X;P}^{n_1}/\mathcal{I}_{Z;P} = \mathfrak{M}_{X;P} \cdot (\mathfrak{M}_{X;P}^{n_1}/\mathcal{I}_{Z;P}).$$

So Nakayama's Lemma implies $\mathcal{I}_{Z;P} = \mathfrak{M}_{X;P}^{n_1}$.

Let $S = \mathrm{Supp}(\mathcal{I}_1|_{H(n_1) \times X})$ and let $\mathcal{E}(n_1)$ be the scheme-theoretical restriction of $\mathcal{E}'$ to $H(n_1) \times X$. Clearly $S \subset \mathcal{E}(n_1)$ is a union of irreducible components of $\mathcal{E}(n_1)$. Let $\tilde{\mathcal{E}}(n_1)$ be the scheme-theoretical closure of $\mathcal{E}(n_1) \cap (H(n_1) \times X \setminus S)$ in $H(n_1) \times X$. Since $\tilde{\mathcal{E}}(n_1)$ is a closed subscheme of $\mathcal{E}(n_1)$ and $\mathcal{O}_{\mathcal{E}(n_1)}$ is a coherent $\mathcal{O}_{H(n_1)}$ -module ($\mathcal{E}(n_1) \rightarrow H(n_1)$ is a projective morphism with finite fibres, so it is a finite morphism), $\mathcal{O}_{\tilde{\mathcal{E}}(n_1)}$ is a coherent $\mathcal{O}_{H(n_1)}$ -module. For a general closed point $(Z; P)$ on some irreducible component of $H(n_1)$ one has

$$\deg([\tilde{\mathcal{E}}(n_1)](Z; P)) = \sum (l(\mathcal{O}_{Z;Q}) : Q \in X(k) \setminus \{P\}) = n - \binom{n_1 + d - 1}{d}.$$

So, because of semicontinuity (see again [6, pp. 1–10]), we obtain for each $(Z; P) \in [H(n_1)](k)$ that $\deg([\tilde{\mathcal{E}}(n_1)](Z; P)) \geq n - \binom{n_1 + d - 1}{d} := n'$.

Lemma 6. For each $(Z; P) \in [H(n_1)](k)$ one has

$$\deg([\tilde{\mathcal{E}}(n_1)](Z; P)) = n'.$$

Proof. Suppose for some $(Z; P) \in [H(n_1)](k)$ that one has $\deg([\tilde{\mathcal{E}}(n_1)](Z; P)) > n'$. Since $[\tilde{\mathcal{E}}(n_1)](Z; P) \subset [\mathcal{E}(n_1)](Z; P)$ and $\sum (l(\mathcal{O}_{Z;Q}) : Q \in X(k) \setminus \{P\}) = n'$, this implies $(Z; P; P) \in [\tilde{\mathcal{E}}(n_1)](k)$, so the closure of some irreducible component of $\mathcal{E}(n_1) \cap (H(n_1) \times X \setminus S)$ intersects S . Let T be the local ring of some irreducible component of $H(n_1)$ containing $(Z; P)$ at $(Z; P)$. We write $\circ$ for the closed point of $\mathrm{Spec}(T)$ and η for the generic point of $\mathrm{Spec}(T)$. The closed subscheme $\mathcal{E}(n_1)$ gives rise to a finite morphism $q: \mathcal{Z} \rightarrow \mathrm{Spec}(T)$ with $\mathcal{Z}$ a closed subscheme of $\mathrm{Spec}(T) \times X$ and $\mathcal{Z}(\circ) = Z \subset X$. Moreover, S gives rise to a section $\mathcal{S}$ of q with $\mathcal{S}(\circ) = P$ and $\mathcal{I}_{\mathcal{Z}(\eta); \mathcal{S}(\eta)} = \mathfrak{M}_{X \times \mathrm{Spec}(k(\eta)); \mathcal{S}(\eta)}^{n_1}$. Moreover, the scheme-theoretical closure of $\mathcal{Z}(\eta) \setminus \mathcal{S}(\eta)$ in $\mathrm{Spec}(T) \times X$ contains $P = \mathcal{S}(\circ)$. Using a suited covering of $\mathrm{Spec}(T)$ we can assume that each point of $\mathcal{Z}(\eta) \setminus \mathcal{S}(\eta)$ is $k(\eta)$ -rational. There exists $V \in \mathcal{Z}(\eta) \setminus \mathcal{S}(\eta)$, giving rise to a section $\mathcal{V}$ of q (because the closure is contained in $\mathcal{Z}$), with $\mathcal{V}(\circ) = P = \mathcal{S}(\circ)$. Let $\mathcal{Z}'$ be the scheme-theoretical closure of the restriction

of $\mathcal{Z}$ to the local rings of $\mathcal{S}(\eta) \cup \mathcal{V}(\eta)$ in $\text{Spec}(T) \times X$. Since $\mathcal{O}_{\mathcal{Z}'}$ is a quotient sheaf of $\mathcal{O}_{\mathcal{Z}}$, we know that $\mathcal{O}_{\mathcal{Z}'}$ is coherent as a $\mathcal{O}_{\text{Spec}(T)}$ -module. Once more, from [6, pp. 1–10], we obtain

$$\deg(\mathcal{Z}'(\circ)) \geq \deg(\mathcal{Z}'(\eta)) \geq \binom{n_1 + d - 1}{d} + 1.$$

But $\text{Supp}(\mathcal{Z}'(\circ)) = \{P\}$, so $l(\mathcal{O}_{\mathcal{Z}'(\circ); P}) \geq \binom{n_1 + d - 1}{d} + 1$. But since $\mathcal{Z}' \subset \mathcal{Z}$ as schemes and $l(\mathcal{O}_{\mathcal{Z}(\circ); P}) = l(\mathcal{O}_{\mathcal{Z}; P}) = \binom{n_1 + d - 1}{d}$, we obtain a contradiction.

Let $\tilde{H}$ be an irreducible component of $H(n_1)$. Let $\tilde{\mathcal{E}}$ be the restriction of $\tilde{\mathcal{E}}(n_1)$ to $\tilde{H} \times X$. We obtained for each $x \in \tilde{H}$ that $\deg(\tilde{\mathcal{E}}(x)) = n'$ (we proved it only for closed points, but then semicontinuity, i.e., again [6, pp. 1–10], implies the claim for all points). From [3, Theorem 9.9, p. 261], it follows that $\tilde{\mathcal{E}}(n_1) \rightarrow \tilde{H}$ is a flat family of 0-dimensional subschemes of X of degree n' . This gives rise to natural morphisms $\tilde{\phi}: \tilde{H} \rightarrow \text{Hilb}^{n'}(X)$ and those morphisms glue together to give a morphism $\phi: H(n_1) \rightarrow \text{Hilb}^{n'}(X)$. This map is described as follows on sets: $\phi(Z; P) = Z \cap (X \setminus \{P\}) \subset X$. Clearly $Z \in H(\mathbf{n})$ if and only if there exists $P \in X(k)$ with $(Z; P) \in H(n_1)$ and $\phi(Z; P) \in H(\mathbf{n}')$ (we abbreviate $(n_2; \dots; n_t)$ to $\mathbf{n}'$). Using $\text{pr}_1: H \times X \rightarrow H$ we obtain $H(\mathbf{n}) = \text{pr}_1(\phi^{-1}(H(\mathbf{n}')))$. This implies that $H(\mathbf{n})$ is locally closed in $H(k)$. Indeed, the induction hypothesis gives us $H(\mathbf{n}') = V'_1 \setminus V'_2$ for some closed subsets V'_1 and V'_2 of $\text{Hilb}^{n'}(X)$. Let V_1 (resp. V_2) be the closure of $\phi^{-1}(V'_1)$ (resp. $\phi^{-1}(V'_2)$) in $(H \times X)(k)$. Then $\text{pr}_1(V_1)$ and $\text{pr}_1(V_2)$ are closed subsets of $H(k)$. Let $V = \text{pr}_1(H'(n_1)) \cap \text{pr}_1(V_1)$ and let $V' = \text{pr}_1(H''(n_1)) \cup \text{pr}_1(V_2)$.

Claim. $H(\mathbf{n}) = V \setminus V'$.

Proof. Let $Z \in H(\mathbf{n})$. Take $P \in X(k)$ with $\mathcal{J}_{Z; P} = \mathfrak{M}_{X; P}^{n_1}$. Then $(Z; P) \in H(n_1)$. Let $Z' = Z \cap (X \setminus \{P\})$. Then $Z' \in H(\mathbf{n}')$ and $Z' = \phi(Z; P)$. This implies $Z \in V$.

Assume $Z \in V'$. Since $Z \notin H''(n_1)$, this implies $Z \in \text{pr}_1(V_2)$. Hence there exists $P' \in X(k)$ with $(Z; P') \in V_2$. Since $V_2 \subset H'(n_1)$, we have $\mathcal{J}_{Z; P'} \subset \mathfrak{M}_{X; P'}^{n_1}$, and since $Z \notin \text{pr}_1(H''(n_1))$, we have $(Z; P') \in V_2 \cap H(n_1) = \phi^{-1}(V'_2)$ and $\mathcal{J}_{Z; P'} = \mathfrak{M}_{X; P'}^{n_1}$. But since $Z \in H(\mathbf{n})$ one necessarily has $\phi(Z; P') \in H(\mathbf{n}')$, so $\phi(Z; P') \notin V'_2$, a contradiction. This proves $H(\mathbf{n}) \subset V \setminus V'$.

Conversely, let $Z \in V \setminus V'$. Since $Z \in V \subset \text{pr}_1(H'(n_1)) \cap \text{pr}_1(V_1)$, there exists $P \in X(k)$ with $\mathcal{J}_{Z; P} \subset \mathfrak{M}_{X; P}^{n_1}$ and $(Z; P) \in V_1$. Since $Z \notin V' \supset \text{pr}_1(H''(n_1))$, this implies $\mathcal{J}_{Z; P} = \mathfrak{M}_{X; P}^{n_1}$, hence $(Z; P) \in H(n_1) \cap V_1 = \phi^{-1}(V'_1)$, so $\phi(Z; P) \in V'_1$. Since $Z \notin V' \supset \text{pr}_1(V_2)$, we have $\phi(Z; P) \notin V'_2$, so $\phi(Z; P) \in V'_1 \setminus V'_2 = H(\mathbf{n}')$. This implies $Z \in H(\mathbf{n})$.

This completes the proof of the claim and the theorem.

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