GENERALIZED SPECTRUM AND A PROBLEM OF APOSTOL

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We give a complete answer to a question of Apostol concerning the asymptotic behaviour of the reduced minimum modulus of a bounded operator on Hilbert space.

Let H be a Hilbert space and $B(H)$ the algebra of bounded operators of H . For $A \in B(H)$, we denote by $N(A)$ and $R(A)$, respectively, the kernel and the range of A .

Define the reduced minimum modulus $\gamma(A)$ of A by setting

$$\gamma(A) = \inf\{\|Au\|; u \in N(A)^\perp \text{ and } \|u\| = 1\}$$

if $A \neq 0$; in this paper it will be convenient to take $\gamma(A) = 0$ if $A = 0$ (see Apostol [1]). It is well known (see [2, Chapter IV, §5]) that $\gamma(A) = \gamma(A^*)$ and that $\gamma(A) > 0 \Leftrightarrow R(A)$ is closed.

Following Apostol, put

$$\sigma_\gamma(A) = \left\{ \mu \in \mathbb{C}; \lim_{\lambda \rightarrow \mu} \gamma(A - \lambda I) = 0 \right\}$$

and

$$\rho_\gamma(A) = \mathbb{C} \setminus \sigma_\gamma(A).$$

In [3, 4] we give a generalization of the spectrum in which invertibility is replaced by the existence of an “analytic generalized resolvent”: we put

$$\text{reg}(A) = \{\lambda \in \mathbb{C}; \forall n \geq 1, N(A - \lambda I)^n \subseteq R(A - \lambda I) \text{ closed}\}.$$

It is familiar that $N(A - \lambda I)^n \subseteq R(A - \lambda I) \Leftrightarrow N(A - \lambda I) \subseteq R(A - \lambda I)^n$, $\text{reg}(A)$ is open, and $\rho(A) \subseteq \text{reg}(A)$. When $0 \in \text{reg}(A)$ we call A “regular.” The generalized spectrum of A , denoted by $\sigma_g(A)$, is the complement of $\text{reg}(A)$ in $\mathbb{C}$ and has properties analogous to the classical spectrum $\sigma(A)$ of A (see [4]). Also [4, Corollaire 2.4] shows that $\sigma_g(A) = \sigma_\gamma(A)$.

Theorem 1. *Let $A \in B(H)$. Then the following conditions are equivalent:*

- (1) $A = A' \oplus Q$, where Q is nilpotent and $0 \in \text{reg}(A')$.
- (2) 0 is an isolated point of $\sigma_g(A)$ and $\lim_{n \rightarrow \infty} \gamma(A^n)^{1/n}$ exists and is strictly positive.

Received by the editors June 24, 1991 and, in revised form, November 5, 1991.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 47A53, 47A55, 47B05.

Key words and phrases. Generalized spectrum, reduced minimum modulus.

Proof. (1) $\Rightarrow$ (2) If $A = A' \oplus Q$ (topological direct sum, not necessarily orthogonal), then there exist closed subspaces $M, N \subset H$, which are A -invariant such that $H = M \oplus N$, $A' = A|_M$, and $Q = A|_N$. Since the operator Q is nilpotent, there is $d \in \mathbb{N}^*$ such that $Q^d = 0$. Therefore we have $N \subset N(A^d)$. To begin we show that for all $\lambda \neq 0$, $N(A - \lambda I) \subset M$. Let $u \in N(A - \lambda I)$. Then $u = v + w$, where $v \in M$ and $w \in N$. Furthermore,

$$0 = (A - \lambda I)u = (A - \lambda I)v + (A - \lambda I)w$$

implies

$$(A - \lambda I)v = -(A - \lambda I)w \in M \cap N = \{0\}.$$

So we have $(A - \lambda I)w = 0$ and $Aw = \lambda w$. Hence $\forall n \geq 0$, $A^n w = \lambda^n w$.

So $w \in N \subseteq N(A^d) \Rightarrow 0 = A^d w = \lambda^d w$. Since $\lambda \neq 0$, it follows that $w = 0$ and therefore $u = v \in M$. Hence $N(A - \lambda I) \subseteq M$ and

$$N(A - \lambda I) = N((A - \lambda I)|_M).$$

Since the operator $A|_M$ is regular, there is a neighbourhood $\mathcal{U}$ of $0 \in \mathbb{C}$ such that $\forall \lambda \in \mathcal{U}$, $(A - \lambda I)|_M$ is regular (since $\text{reg}(A|_M)$ is open). So $R((A - \lambda I)|_M)$ is closed $\forall \lambda \in \mathcal{U}$. Furthermore, $\forall \lambda \neq 0$, $N(A - \lambda I) = N((A - \lambda I)|_M)$. This implies that for all $\lambda \in \mathcal{U} - \{0\}$, and for all $n \geq 0$, one has $N((A - \lambda I)^n) = N(((A - \lambda I)|_M)^n) \subseteq R((A - \lambda I)|_M)$. Therefore, $\forall \lambda \in \mathcal{U} - \{0\}$, $\forall n \geq 0$, $N((A - \lambda I)^n) \subset R(A - \lambda I)$.

Now we prove that $R(A - \lambda I)$ is closed for all $\lambda \in \mathcal{U} - \{0\}$. The operator $A|_N$ being nilpotent, for all $\lambda \neq 0$, $(A - \lambda I)|_N$ is invertible in $B(N)$, hence $N = R((A - \lambda I)|_N)$. Consequently, for all $\lambda \in \mathcal{U} - \{0\}$, we have that $R(A - \lambda I) = R((A - \lambda I)|_M) \oplus N$. This implies that $R(A - \lambda I)$, as a sum of two closed subspaces of M and N , is closed. Hence $\mathcal{U} - \{0\} \subseteq \text{reg}(A)$, where such 0 is isolated in $\sigma_g(A)$.

Furthermore, [4, Théorème 3.1, Corollaire 3.9] show that $\lim_{n \rightarrow \infty} \gamma(A^n)^{1/n}$ exists and is strictly positive.

(2) $\Rightarrow$ (1) follows [1, Proposition 3.3]. $\square$

Remark. The implication (1) $\Rightarrow$ (2) of Theorem 1 implies that in [1, Proposition 3.3], $\sigma = \{0\}$. Furthermore, [1, Theorem 3.5] implies also that $\sigma = \{0\}$.

Proposition 2. Let $A \in B(H)$ such that $0 \in \partial\sigma(A)$, the boundary of $\sigma(A)$. Then $\lim_{n \rightarrow \infty} \gamma(A^n)^{1/n} > 0$ implies that 0 is isolated in $\sigma(A)$.

Proof. We remark that $\partial\sigma(A) \subseteq \sigma_g(A)$ (see [4, Corollary 2.2]). Using [1, Theorem 3.5], we see that 0 is isolated in $\sigma_g(A)$. Hence there is a neighbourhood $\mathcal{U}$ of $0 \in \mathbb{C}$ such that $\mathcal{U} - \{0\} \subseteq \text{reg}(A)$. Since $0 \in \partial\sigma(A)$, it follows that $(\mathcal{U} - \{0\}) \cap \rho(A) \neq \emptyset$. Consequently, $\mathcal{U} - \{0\} \subseteq \rho(A)$ (see [4, Corollary 2.3]). The proposition is proved. $\square$

Corollary 3. If $0 \in \partial\sigma(A)$, then

$$0 \text{ not isolated in } \sigma(A) \Rightarrow \overline{\lim_{n \rightarrow \infty} \gamma(A^n)^{1/n}} = 0.$$

Proof. Direct consequence of Proposition 2. $\square$

Remark. (1) If 0 is isolated in $\sigma(A)$ then both cases are possible.

For example, if A is selfadjoint and not invertible, with closed range $R(A)$, then $H = N(A) \oplus R(A)$ and for all $n > 0$, we have $\gamma(A^n) = \gamma(A)^n$. It is clear that 0 is isolated in $\sigma(A)$, and since $R(A)$ is closed, we have $\gamma(A) > 0$.

If instead A is nilpotent then $\lim_{n \rightarrow \infty} \gamma(A^n)^{1/n} = 0$ since $\gamma(A) = 0$ if $A = 0$.

(2) Corollary 3 and the preceding examples give a complete answer to Question 1 posed by Apostol [1, p. 289].

REFERENCES

1. C. Apostol, *The reduced minimum modulus*, Michigan Math. J. **32** (1985), 279–294.
2. T. Kato, *Perturbation theory for linear operators*, Springer, New York, 1966.
3. M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasgow Math. J. **29** (1987), 159–175.
4. —, *Résolvant généralisé et théorie spectrale*, J. Operator Theory **21** (1989), 69–105.

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