SOME SPECIAL LIMITS OF SCHOTTKY GROUPS

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ABSTRACT. We show that every extended Schottky group G which uniformizes a compact Riemann surface S is a geometric limit of Schottky groups G_n which also uniformize S . That is, every element $g \in G$ is the limit of elements $g_n \in G_n$.

Let Γ be a Fuchsian group covering a compact Riemann surface S , and let (f, χ) be a projective structure on S . Suppose that f is a covering map and that $\chi(\Gamma)$ is an extended Schottky group. (Definitions will be given in the text. For the moment, we note that our definition of extended Schottky group does *not* include Schottky groups.) In this paper, we will show that there exists a sequence (f_n, χ_n) of projective structures on S with the following properties: (i) $\chi_n(\Gamma)$ is a Schottky group, and (ii) (f_n, χ_n) converges to (f, χ) . The precise statement, Theorem 1, and proof of this result are given in §3. In §4 we prove an analogue of Theorem 1 for compact surfaces of genus 1.

We note that the homomorphisms χ_n converge algebraically to χ . In our proofs, we exhibit cyclic subgroups of $\chi_n(\Gamma)$ which have a rank 2 parabolic subgroup of $\chi(\Gamma)$ as a geometric limit (Lemma 1 and Theorem 2). This reproduces, within the context of algebraic convergence of projective structures on a fixed Riemann surface, the basic example of a geometric limit given by Jorgensen and Thurston (see [9] for details).

1. PRELIMINARIES

Throughout, Γ will denote a torsion-free Fuchsian group acting on the upper half plane U and covering a compact surface U/Γ . We denote the space of holomorphic quadratic differentials (defined on U) for Γ by $B_2(\Gamma)$ and equip $B_2(\Gamma)$ with the norm $\|\varphi\| = \sup_{z \in U} \{|\varphi(z)|(2\operatorname{Im} z)^2\}$.

For every $\varphi \in B_2(\Gamma)$ there exists a meromorphic local homeomorphism $f_\varphi = f$, where $f : U \rightarrow \widehat{\mathbb{C}}$ and $S(f) = \varphi$; here $S(f)$ is the *Schwarzian derivative* of f given by

$$S(f) = (f''/f')' - \frac{1}{2}(f''/f')^2.$$

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The function f determines a homomorphism $\chi_\varphi : \Gamma \rightarrow \text{PSL}(2, \mathbb{C})$ with the defining property $f \circ \gamma(z) = \chi_\varphi(\gamma) \circ f(z)$ for all $\gamma \in \Gamma$, $z \in U$.

The pair (f, χ_φ) is called a *projective structure* on U/Γ . In general, we will require that f be normalized by the conditions $f(i) = 0$, $f'(i) = 1$, $f''(i) = 0$. With this normalization, there is a one-to-one correspondence between projective structures on U/Γ and elements of $B_2(\Gamma)$ (see [5, 7] for details). We will restrict our attention in this paper to the subset $C_2(\Gamma) \subset B_2(\Gamma)$ defined by

$$C_2(\Gamma) = \{\varphi \in B_2(\Gamma) : f \text{ is a covering map}\}.$$

It was shown by Kra-Maskit [7] that $C_2(\Gamma)$ is a closed and bounded subset of $B_2(\Gamma)$. We note that, for $\varphi \in C_2(\Gamma)$, $\chi_\varphi(\Gamma)$ is a Kleinian group with $f(U)$ an invariant component (see [5]).

2. UNIFORMIZATIONS OF SCHOTTKY TYPE

By a *quadrilateral* Q in the extended complex plane $\hat{\mathbb{C}}$, we will mean a Jordan curve $Q \subset \hat{\mathbb{C}}$ on which four distinct vertices have been chosen so as to yield opposite pairs of sides a, a' and b, b' . Let $\alpha, \beta \in \text{PSL}(2, \mathbb{C})$ be parabolic transformations with a common fixed point contained in one of the complementary components of Q and suppose that α and β identify opposite sides of Q so that $J = \langle \alpha, \beta \rangle$ is a rank 2 parabolic group. We will say that J is a *parabolic group associated to* Q .

Let $Q_1, \dots, Q_p, C_{p+1}, \dots, C_{p+q}, C'_{p+1}, \dots, C'_{p+q}$ be a collection of mutually disjoint Jordan curves in $\hat{\mathbb{C}}$ that bound a connected set F . Suppose Q_i , $i = 1, \dots, p$, are quadrilaterals with opposite pairs of sides a_i, a'_i and b_i, b'_i and associated parabolic groups $J_i = \langle \alpha_i, \beta_i \rangle$, which have their fixed points in the interior of Q_i (that component not intersecting F). Let $\alpha_i(b_i) = b'_i$, $\beta_i(a_i) = a'_i$.

Furthermore, suppose γ_j , $j = p+1, \dots, p+q$, are loxodromic transformations which map the interior of C_j onto the exterior of C'_j . Denote by G the free product $J_1 \times \dots \times J_p \times \langle \gamma_{p+1} \rangle \times \dots \times \langle \gamma_{p+q} \rangle$. We will say that G is a group of *Schottky type* if $p+q > 1$. A group of Schottky type is an *extended Schottky group* if $p > 0$; it is a *Schottky group* if $p = 0$. (We note that this terminology is not standard.) It is well known that G is a Kleinian group with a connected region of discontinuity $\Omega(G)$.

The compact surface $\Omega(G)/G$ has genus $p+q$ and the natural projection $\pi : \Omega(G) \rightarrow \Omega(G)/G$ is an unbranched covering map. The images $\pi(a_i)$ and $\pi(b_i)$ correspond to simple closed curves which are homologically independent; $\pi(a_i)$ goes around the i th hole, and $\pi(b_i)$ goes around the i th handle. $\pi(C_j)$ is a simple closed curve which goes around the j th hole.

3. STATEMENT AND PROOF OF MAIN RESULT

Given a Kleinian group G with an invariant component $\Omega(G)$, let $h : U \rightarrow \Omega(G)$ be a holomorphic universal covering map. The *Fuchsian equivalent* Γ of G is defined by

$$\Gamma = \{\gamma \in \text{PSL}(2, \mathbb{C}) : \exists g \in G \text{ with } h \circ \gamma = g \circ h\}.$$

Define $\chi : \Gamma \rightarrow G$ by $\chi(\gamma) = g$ whenever $h \circ \gamma = g \circ h$. It follows that $U/\Gamma = \Omega(G)/G$, that (h, χ) is a projective structure on Γ , and that $S(h) \in C_2(\Gamma)$.

Precomposing h with a suitable Möbius transformation T , and conjugating G by T , we may assure that h is normalized.

Theorem 1. *Let $\varphi \in C_2(\Gamma)$ with $\chi_\varphi(\Gamma)$ an extended Schottky group. Then there exists a sequence $\varphi_n \in C_2(\Gamma)$, $\varphi_n \rightarrow \varphi$, with $\chi_{\varphi_n}(\Gamma)$ a Schottky group.*

In order to prove the theorem we will use a preliminary construction and four lemmas. Let (h, χ) be a projective structure on U/Γ , where Γ is the Fuchsian model of an extended Schottky group $\chi(\Gamma)$. The structure (h, χ) will remain fixed from now on. As described in §2, we have associated to $\chi(\Gamma)$ quadrilaterals $Q_i \subset \hat{\mathbb{C}}$ and Jordan curves C_j, C'_j , $i = 1, \dots, p$, $j = p + 1, \dots, p + q$. (We will avail ourselves of all the notation established in §2.) Let $z_0 \in \Omega(\chi(\Gamma))/\chi(\Gamma) = S$ and choose simple loops $A_i, B_i \in \pi_1(S, z_0) = \pi_1(S)$ homologous to $\pi(a_i)$ and $\pi(b_j)$, respectively. Choose $A_j \in \pi_1(S)$ to be a simple loop homologous to $\pi(C_j)$. It is well known (see, e.g., Hejhal [2]) that we may choose simple loops $B_j \in \pi_1(S)$ so that $\{A_i, B_i; A_j, B_j\}$ is a canonical set of generators for $\pi_1(S)$. That is, $\prod_{i=1}^p [A_i, B_i] \prod_{j=p+1}^{p+q} [A_j, B_j] = \text{id}$. (Here, $[A, B]$ denotes the commutator $ABA^{-1}B^{-1}$.)

Identifying $\pi_1(S)$ with Γ so that we have $\chi(A_i) = \alpha_i$, $\chi(B_i) = \beta_i$, and $\chi(B_j) = \gamma_j$, we also have from uniformization theory (see [8]) $\ker \chi = \langle\langle A_i B_i A_i^{-1} B_i^{-1}; A_j \rangle\rangle = N$. Here, and elsewhere, $\langle\langle \cdot \rangle\rangle$ denotes the normal subgroup of Γ generated by the given arguments.

We now construct a sequence of projective structures on U/Γ . Observe that, for each $n \geq 0$, $\{A_i B_i^n, B_i; A_j, B_j\}$ is a canonical set of generators for Γ . Let $N_n = \langle\langle A_i B_i^n; A_j \rangle\rangle$. It is a classical result (see [2]) that N_n determines a projective structure (f_n, χ_n) on U/Γ , with f_n normalized, where $G_n = \chi_n(\Gamma) = \Gamma/N_n$ is a Schottky group. The region of discontinuity of G_n is $f_n(U)$ and $\chi_n(B_i), \chi_n(B_j)$ are free generators for G_n . Moreover, $f_n(U)/G_n = U/\Gamma$, Γ is the Fuchsian model of G_n , and $\ker \chi_n = N_n$. We note that the groups N_n are distinct. This last assertion is easily verified by passing to $\overline{N}_n$, the projection of N_n in homology, and noting that the groups $\overline{N}_n$ are distinct. It follows that the structures (f_n, χ_n) are distinct (see [2, 8]).

Now $S(f_n) \in C_2(\Gamma)$, and, applying the results of [7], we choose a subsequence (f_n, χ_n) and a projective structure (f, χ^*) on Γ such that $S(f_n) \rightarrow S(f)$. It follows that $f_n \rightarrow f$ uniformly on compact subsets of U , and $\chi_n \rightarrow \chi^*$ algebraically. Note that f is normalized and $S(f) \in C_2(\Gamma)$. Our objective is to show that $\ker \chi^* = \ker \chi$ (Lemma 4).

It is a consequence of a result of Jorgensen-Klein [4] (see also, Jorgensen [3] or Kra [6]) that the map $\theta_n : \chi^*(\Gamma) \rightarrow \chi_n(\Gamma)$ which sends $\chi^*(\gamma)$ to $\chi_n(\gamma)$, $\gamma \in \Gamma$, is a homomorphism for almost all n . Thus, $\chi_n = \theta_n \circ \chi^*$, and one has

$$(3.1) \quad \ker \chi^* \subset \ker \chi_n.$$

Although we will not need the full strength of the next lemma, we include it for clarity.

Lemma 1. *For a fixed i , $i = 1, \dots, p$, $\chi^*(A_i)$, $\chi^*(B_i)$ are parabolic and generate a rank 2 parabolic group.*

Proof. For simplicity, we drop the subscript i . For each n , $\chi_n(AB^n) = \text{id}$ and $\chi_n(A) = \chi_n(B^{-n})$. Since $\chi_n(A) \rightarrow \chi^*(A)$, it follows that $\chi_n(B^{-n}) \rightarrow \chi^*(A)$. In particular, $\chi_n(B^{-n})$ converges.

Since $\chi_n(B) \neq \text{id}$ for all n and $\chi_n(\Gamma)$ is a free group, we have that $\chi_n(B^m) \neq \text{id}$ for all $m \neq 0$. Thus, applying (3.1), $\chi^*(B^m) \neq \text{id}$ and $\chi^*(B)$ is neither elliptic nor trivial. Similarly, since $\chi_n(A) = \chi_n(B^{-n}) \neq \text{id}$, (3.1) yields that $\chi^*(A)$ is neither trivial nor elliptic.

Hence, $\chi^*(B)$ is either loxodromic or parabolic. Suppose it is loxodromic. Letting $\lambda_n, \lambda \in \mathbb{C}$, with $|\lambda_n|, |\lambda| > 1$, be the multipliers of $\chi_n(B)$ and $\chi^*(B)$, respectively, it follows easily that $|\lambda_n| \rightarrow |\lambda|$; but, since $\chi_n(B^{-n})$ converges, we also have that $|\lambda_n|^n$ converges. This contradiction establishes that $\chi^*(B)$ is parabolic. Since the fixed points of $\chi_n(B^{-n})$ coincide with the fixed points of $\chi_n(B)$, they must also converge to the fixed point of $\chi^*(B)$. Hence, $\lim \chi_n(B^{-n}) = \chi^*(A)$ is parabolic and $\chi^*(A), \chi^*(B)$ have a common fixed point.

To prove that $\chi^*(A), \chi^*(B)$ generate a rank 2 parabolic group, it suffices to show that $\chi^*(A^s) \neq \chi^*(B^t)$ for all integers $s, t \neq 0$. Now, for fixed $s, t \neq 0$, $t = -ns$ is satisfied by at most one value of n . Hence, for almost all n , $\chi_n(A^s B^{-t}) = \chi_n(B^{-ns} B^{-t}) \neq \text{id}$. Applying (3.1), we have that $\chi^*(A^s B^{-t}) \neq \text{id}$, and the lemma follows.

Given r a fixed integer, $r = 1, \dots, p+q$, the nontrivial words $w(A_r, B_r) \in \Gamma$ in the generators A_r, B_r may be written as

$$(3.2) \quad w(A_r, B_r) = \prod_{k=1}^s A_r^{m_k} B_r^{n_k}.$$

Here, $s \geq 1$ and at least one of the integers $m_1, n_1 \neq 0$. If $s > 1$, then $n_1 \neq 0, m_s \neq 0$, and $m_k, n_k \neq 0$, for $1 < k < s$. The length of w , denoted by $|w|$, is s . Define $w^A = \sum_{k=1}^s m_k$ and $w^B = \sum_{k=1}^s n_k$. For $r = 1, \dots, p$, we will write $[w] = 0$, if $w^A = w^B = 0$. For $r = p+1, \dots, p+q$, we will write $[w] = 0$, if $w^B = 0$.

Lemma 2. *Let i be a fixed integer, $i = 1, \dots, p$, and $w(A_i, B_i) \in \Gamma$ a nontrivial word with $[w] = 0$. Then $w \in \langle \langle A_i B_i A_i^{-1} B_i^{-1} \rangle \rangle = K_i$.*

Proof. Dropping the subscript i , we write $w = \prod_{k=1}^s A^{m_k} B^{n_k}$ as in (3.2). The proof is by induction on s . First, consider the induced map $\theta : \Gamma \rightarrow \Gamma/K$, and note that $\theta(A), \theta(B)$ commute. Since Γ/K has a representation as a discrete, torsion-free subgroup of $\text{PSL}(2, \mathbb{C})$ (see [8]), we have that all powers of $\theta(A), \theta(B)$ commute. Hence, $A^m B^n A^{-m} B^{-n} \in K$ for all integers m, n . It follows that the lemma is true for $s = 1, 2$.

We assume the lemma holds for all $k < s$ with $s > 2$. After conjugating by B^{n_s} (that is, taking $B^{n_s} w B^{-n_s}$) and reindexing, we assume $n_s = 0$.

Case 1. $m_1 \neq 0$. Since $s > 2$, we write $w = A^{m_1} B^{n_1} A^{m_2} B^{n_2} \dots A^{m_s}$. Thus, $w = B^{n_1} w_1 A^{m_1+m_2} B^{n_2} \dots A^{m_s}$, where $w_1 = B^{-n_1} A^{m_1} B^{n_1} A^{-m_1}$, $w_1 \in K$, and $B^{-n_1} w B^{n_1} = w_1 w_2$, where $w_2 = A^{m_1+m_2} B^{n_2} \dots A^{m_s} B^{n_1}$. Now, $|w_2| = s-1$ and $[w_2] = 0$ and, by the inductive hypothesis, $w_2 \in K$. Hence, $w = B^{n_1} w_1 w_2 B^{-n_1} \in K$.

Case 2. $m_1 = 0$. Here, $w = B^{n_1} A^{m_2} B^{n_2} \dots A^{m_s}$, and letting $w_1 = B^{-n_1} w B^{n_1} = A^{m_2} B^{n_2} \dots A^{m_s} B^{n_1}$, we have $|w_1| = s-1$, $[w_1] = 0$. By the inductive hypothesis, $w_1 \in K$, and it follows that $w \in K$.

Lemma 3. *Let j be a fixed integer, $j = p+1, \dots, p+q$, and $w(A_j, B_j) \in \Gamma$ a nontrivial word with $[w] = 0$. Then $w \in \langle \langle A_j \rangle \rangle = K_j$.*

Proof. Dropping the subscript j , we write $w = \prod_{k=1}^s A^{m_k} B^{n_k}$ and define $\bar{t} = \sum_{k=1}^t n_k$, $t = 1, \dots, s$.

Rewrite $w = A^{m_1} P_1 P_2 \cdots P_{s-1} B^{\bar{s}}$, where $P_t = B^{\bar{t}} A^{m_{t+1}} B^{-\bar{t}} \in K$. Since $\bar{s} = 0$, we have $w \in K$. This concludes the proof of the lemma.

Now consider $w(A_i, B_i; A_j, B_j) \in \Gamma$ an arbitrary, nontrivial word in the generators A_i, B_i, A_j, B_j , $i = 1, \dots, p$, $j = p+1, \dots, p+q$. It may be written

$$(3.3) \quad w = w_1(A_{k_1}, B_{k_1}) \cdots w_r(A_{k_r}, B_{k_r}),$$

where $k_m \in \{1, \dots, p+q\}$, $m = 1, \dots, r$, w_m is a nontrivial word, and $k_m \neq k_{m+1}$ for $m < r$.

Lemma 4. $\ker \chi^* = N$.

Proof. It is clear from Lemma 1 that $\chi_n(A_i)$, $\chi_n(B_i)$ commute; hence, since $\chi_n(A_j) = \text{id}$, we have that $N \subset \ker \chi_n$ for all $n \geq 0$. Thus, $N \subset \ker \chi^*$. To show the opposite inclusion, suppose $w(A_i, B_i; A_j, B_j) \in \Gamma$ is a word with $w \notin N$. Choose $\hat{w} \in \Gamma$ to be a word which is equivalent to $w \pmod{N}$ and contains no subwords lying in N . Then, since $N \subset \ker \chi_n$, $\chi_n(w) = \chi_n(\hat{w})$.

We write $\hat{w} = w_1(A_{k_1}, B_{k_1}) \cdots w_r(A_{k_r}, B_{k_r})$ as in (3.3). Then

$$(3.4) \quad \chi_n(w) = \prod_{m=1}^r \chi_n(w_m(A_{k_m}, B_{k_m})).$$

Since $\chi_n(A_{k_m})$ commutes with $\chi_n(B_{k_m})$, it follows from (3.4) that

$$(3.5) \quad \chi_n(w) = \prod_{m=1}^r \chi_n(A_{k_m}^{m_1}) \chi_n(B_{k_m}^{m_2}),$$

where $m_1 = w_m^A$, $m_2 = w_m^B$. Now $w_m \notin N$, hence $w_m \notin K_{k_m}$. Applying Lemmas 2 and 3, we have that for each m at least one of the integers m_1 or m_2 is not 0.

Since $\chi_n(A_{k_m}) = \chi_n(B_{k_m}^{-n})$ for $k_m = 1, \dots, p$ and $\chi_n(A_{k_m}) = \text{id}$ for $k_m = p+1, \dots, p+q$, we obtain from (3.5) that

$$(3.6) \quad \chi_n(w) = \prod_{m=1}^r \chi_n(B_{k_m}^{-nm_1+m_2}).$$

But $-nm_1 + m_2 = 0$ for at most one value of n . Hence, (3.6) implies that, for almost all n , $\chi_n(w) \neq \text{id}$ (recall that $k_m \neq k_{m+1}$, $m < r$). Applying (3.1), one has that $\chi^*(w) \neq \text{id}$. This concludes the proof of Lemma 4.

Proof of Theorem 1. Let $\varphi \in C_2(\Gamma)$ determine the normalized projective structure $(f_\varphi, \chi_\varphi)$, with $\chi_\varphi(\Gamma)$ an extended Schottky group. Suppose Γ is the Fuchsian equivalent of $\chi_\varphi(\Gamma)$.

As we have just shown, there exists a sequence of projective structures (f_n, χ_n) , with $S(f_n) = \varphi_n \in C_2(\Gamma)$ and $\chi_n(\Gamma)$ a Schottky group, converging to (f, χ^*) with $S(f) \in C_2(\Gamma)$ and $\ker \chi^* = \ker \chi_\varphi$. It is easy to show that the Kleinian group $\chi^*(\Gamma)$ has a connected region of discontinuity (see [7, 8] for a proof). Moreover, it is known that, for groups with a connected region of discontinuity, the conformal structure of the quotient surface plus the

kernel of the homomorphism determine the homomorphism up to conjugation in $\mathrm{PSL}(2, \mathbb{C})$ whenever the kernel is nontrivial (see [7, 8]). Hence, since the kernels of the two homomorphisms agree and f, f_φ are normalized, $\chi^* = \chi_\varphi$. It follows from the theory of projective structures that $S(f) = \varphi$ (see [5]).

Now assume that Γ is not the Fuchsian equivalent of $\chi_\varphi(\Gamma)$. In this situation, there exist a finitely generated Fuchsian group F with $\Gamma \subset F$ and a projective structure $(f_\varphi, \hat{\chi})$ on U/F , where $\hat{\chi}|_\Gamma = \chi_\varphi$, $\hat{\chi}(F) = \chi_\varphi(\Gamma)$, and F is the Fuchsian equivalent of $\hat{\chi}(F)$ (see [7]). As before, choose $\varphi_n \in C_2(F)$ with $f_{\varphi_n} \rightarrow f_\varphi$, $\chi_{\varphi_n} \rightarrow \hat{\chi}$. Let $\chi'_{\varphi_n} = \chi_{\varphi_n}|_\Gamma$. Now $(f_{\varphi_n}, \chi'_{\varphi_n})$ is a projective structure on Γ , and $\chi'_{\varphi_n}(\Gamma)$, being a finitely generated subgroup of a Schottky group, is a Schottky group. Since $\chi'_{\varphi_n} \rightarrow \hat{\chi}|_\Gamma = \chi_\varphi$, the differentials φ_n (note that $\varphi_n \in C_2(\Gamma)$, also) have the required properties.

4. SCHOTTKY STRUCTURES IN GENUS 1

We consider now a compact Riemann surface T of genus 1. It is conformally equivalent to $\mathbb{C}/G$, where G is a rank two parabolic group with generators $A(z) = z + 1$, $B(z) = z + \omega$, and $\omega \in \mathbb{C}$ has positive imaginary part.

In this situation, a projective structure (f, χ) consists of a holomorphic local homeomorphism $f: \mathbb{C} \rightarrow \mathbb{C}$ and a homomorphism $\chi: G \rightarrow A(2, \mathbb{C})$, where $f \circ \gamma = \chi(\gamma) \circ f$. Here, $A(2, \mathbb{C})$ is the *affine* group consisting of all elements of $\mathrm{PSL}(2, \mathbb{C})$ fixing ∞ .

Following Gunning [1], $f''/f' = 2\pi ic$ for some constant $c \in \mathbb{C}$, and the projective structures on T can be parametrized by $\mathbb{C}$. For $c = 0$, we will normalize so that $f_0(z) = z$ and χ_0 is the identity homomorphism. If $c \neq 0$, $f(z) = ae^{2\pi icz} + b$ where $a \neq 0$, and we precompose f by an affine transformation so that $f_c(z) = e^{2\pi icz}$ and $\chi_c(A)(z) = e^{2\pi icz}$, $\chi_c(B)(z) = e^{2\pi ic\omega}z$. (Note that all structures are given by covering maps.)

One sees easily that, for $n > 0$, $c_n = 1/(\omega - n)$ satisfies $|e^{2\pi ic_n}| \neq 1$, with

$$\chi_{c_n}(A)(z) = e^{2\pi i/(\omega-n)}z, \quad \chi_{c_n}(B)(z) = e^{2\pi in/(\omega-n)}z.$$

Thus, $\chi_{c_n}(G) = \langle \chi_{c_n}(A), \chi_{c_n}(B) \rangle$ is a purely loxodromic, discontinuous group generated by $\chi_{c_n}(A)$. Moreover, $(\mathbb{C} - \{0\})/\chi_{c_n}(G) = T$. Extending the definition of §2, we will say that $\chi_{c_n}(G)$ is a Schottky group. Note that $c_n \rightarrow 0$, but χ_{c_n} does not converge algebraically to χ_0 .

In order to obtain algebraic convergence, we renormalize in the following manner: Define $\beta_n \in A(2, \mathbb{C})$ as $\beta_n(z) = z + b_n$, where $b_n = ne^{-2\pi i/(\omega-n)}/2\pi i$. Now, consider the structures (g_n, χ_n) , where $g_n = \beta_n \circ f_{c_n}$, $\chi_n = \beta_n \circ \chi_{c_n} \circ \beta_n^{-1}$. Then

$$\chi_n(A)(z) = e^{2\pi i/(\omega-n)}z - b_n(e^{2\pi i/(\omega-n)} - 1)$$

and

$$\chi_n(B)(z) = e^{2\pi in/(\omega-n)}z - b_n(e^{2\pi in/(\omega-n)} - 1).$$

It is straightforward to verify that $\chi_n(A)(z) \rightarrow z + 1$ and that $\chi_n(B)(z) \rightarrow z + \omega$. Observe that $g_n(z) = e^{2\pi iz/(\omega-n)} + ne^{-2\pi i/(\omega-n)}/2\pi i$ diverges.

In general, for an arbitrary sequence (h_{c_k}, χ_{c_k}) with $c_k \rightarrow 0$, one cannot expect h_{c_k} to converge to a covering map. Note that, necessarily, $h_{c_k}(z) = a_k e^{2\pi ic_k z} + b_k$ and $e^{2\pi ic_k z}$ converges to a constant map.

Nonetheless, we have proven the following analogue to Theorem 1:

Theorem 2. *Let G be a rank 2 parabolic group acting on $\mathbb{C}$. Let (f_0, χ_0) be the identity structure on $\mathbb{C}/G$; that is, $f_0(z) = z$ and χ_0 is the identity homomorphism. Then there exists a sequence of projective structures (f_n, χ_n) on $\mathbb{C}/G$, where $\chi_n(G)$ is a Schottky group and χ_n converges algebraically to χ_0 .*

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