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SIMPLEXES IN RIEMANNIAN MANIFOLDS

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ABSTRACT. Existence of a simplex with prescribed edge lengths in Euclidean, spherical, and hyperbolic spaces was studied recently. A simple sufficient condition of this existence is, roughly speaking, that the lengths do not differ too much. We extend these results to Riemannian n-manifolds M^n. More precisely we consider $m+1$ points $p_0, p_1, \dots, p_m$ in M^n, $m \leq n$, with prescribed mutual distances l_{ij} and establish a condition on the matrix (l_{ij}) under which the points p_i can be selected as freely as in R^n : p_0 is a prescribed point, the shortest path p_0p_1 has a prescribed direction at p_0, the triangle $p_0p_1p_2$ determines a prescribed 2-dimensional direction at p_0, and so on.

1. BASIC DEFINITIONS AND THE THEOREM

Existence of a simplex with prescribed edge lengths in Euclidean, spherical, and hyperbolic spaces was studied in [3]. A simple sufficient condition of this existence established there is, roughly speaking, that the edge lengths do not differ too much, see [3, Theorem 2]. We deal here with $m+1$ points $p_0, p_1, \dots, p_m$ in a Riemannian n-manifold M^n, $m \leq n$, with prescribed mutual distances l_{ij} and establish a condition on the matrix (l_{ij}) under which the points p_i can be selected as freely as in R^n: p_0 is a prescribed point, the shortest path p_0p_1 has a prescribed direction at p_0, the triangle $p_0p_1p_2$ determines a prescribed 2-dimensional direction at p_0, and so on. Our result however does not guarantee uniqueness of the points p_i (see more on that at the ends of parts A and I of §3). Note that the desired points p_i may not exist even though all the distances l_{ij} are equal and the manifold M^n is complete, noncompact, and expanding in the following sense: there exists a point $w \in M^n$ and a constant $c > 0$ such that for any triangle awb with $wa = wb$, one has $ab > c \cdot wa \cdot \angle awb$ where $\angle$ means angle. An appropriate example for four points in M^3 can be constructed as follows. Let M^2 be a narrow right circular cone. Its vertex v can be smoothed out later for regularity. Put $M^3 = M^2 \times R$. One can check that M^3 is expanding if the point $(v, 0)$ is chosen as the point w. Prescribe $l_{ij} = 1$.

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Select points q_0, q_1, and $q_2 \in M^2$ on a circumference centered at v such that their mutual distances in M^2 all are unit. Now put $p_0 = (q_0, 0)$, $p_1 = (q_1, 0)$, and $p_2 = (q_2, 0)$. Obviously $l_{01} = l_{02} = l_{12} = 1$. By symmetry, the last point p_3 should be of the form (v, h), $h \in R$. Then $l_{30} = l_{31} = l_{32} = (r^2 - h^2)^{1/2}$ where r is the radius of the above circumference. When the cone M^2 is sufficiently narrow, one has $r > 1$. Then $l_{30} = l_{31} = l_{32} > 1$ and hence the desired point p_3 does not exist when p_0, p_1, and p_2 are selected as above.

There is another subtle difference in this area between M^n and Euclidean, hyperbolic, or spherical n-space X_k^n of curvature k. Consider, say, a tetrahedron in X_k^n. If the directions of the three edges coming from its vertex are coplanar then the same is true of each other vertex of the tetrahedron. This is not so in M^n even for small tetrahedra.

By k-plane, we will mean X_k^2. The sphere X_1^n will often be denoted by S^n.

The notation xy will be used for a geodesic with ends x, y for its length and for the distance between x and y. The meaning will be specified in cases of possible confusions.

A set $C \subset M^n$ is called *convex* if for each two points in C there exists a unique shortest path in M^n connecting these points and this path (which is a geodesic) belongs to C.

Let (x_{ij}) be a matrix with $x_{ii} = 0$, $x_{ij} = x_{ji} > 0$, $i, j = r, r+1, \dots, r+s$. (We will encounter cases $r = 0$ and $r = 1$.) Such a matrix will be called *allowable.* Let $q_r, q_{r+1}, \dots, q_{r+s}$ be $s+1$ points in a metric space Y with the mutual distances $q_iq_j = x_{ij}$. The set of these $s+1$ points will be called a *realization* of (x_{ij}) in Y and often written down as $q_rq_{r+1}\cdots q_{r+s}$. Suppose $Y = X_k^n$ with $n \geq s$. In case $k > 0$, assume also that the points q_i lie in an open semisphere of X_k^n. If their convex hull is a nondegenerate s-simplex then we say that the matrix (x_{ij}) and its realization are *nondegenerate* in X_k^n.

Let M^n, $n \geq 2$, be a regular Riemannian manifold and let $e_1, e_2, \dots, e_m$, $2 \leq m \leq n$, be pairwise orthogonal unit vectors at a point $p \in M^n$. The set $\{e_1, e_2, \dots, e_m\}$ will be called a *frame* at p. Suppose that an allowable matrix (l_{ij}), $i, j = 0, 1, \dots, m$, has a realization $p_0p_1\cdots p_m$ in M^n such that, for each pair p_i, p_j, the manifold M^n contains a unique shortest geodesic p_ip_j of the length l_{ij} and the following conditions hold.

(0) $p_0 = p$.
(1) The direction of the segment p_0p_1 is e_1.
(2) The direction of the segment p_0p_2 is coplanar with e_1 and e_2 and forms with e_2 an angle $< \pi/2$.
(3) The direction of the segment p_0p_3 is coplanar with e_1, e_2, and e_3 and forms with e_3 an angle $< \pi/2$.
....
(m) The direction of the segment p_0p_m is coplanar with $e_1, e_2, \dots, e_m$ and forms with e_m an angle $< \pi/2$.

We will say then that the realization $p_0p_1\cdots p_m$ of (l_{ij}) *fits the frame* $\{e_1, e_2, \dots, e_m\}$ *at* p.

Theorem. *Let M^n, $n \geq 2$, be a regular Riemannian n-manifold, not necessarily complete. Let $p \in M^n$, $r > 0$ be less than or equal to the convexity radius at p (see* [4, §5.2] *for the definition), let k' and k'' be finite lower and upper bounds of the sectional curvature in the r-neighbourhood $N_r(p)$ of p, and let*

$\{e_1, e_2, \dots, e_m\}$, $2 \le m \le n$, be a frame at p. Suppose that an allowable matrix (l_{ij}), $i, j = 0, 1, \dots, m$, satisfies the following conditions:

(i) $l_{0i} < r$, $i = 1, 2, \dots, m$.

(ii) *For each pair of distinct i and j different from 0, there exists a nondegenerate triangle on the k'-plane (k''-plane) with side lengths l_{0i}, l_{0j}, and l_{ij}.*

$$(1) \qquad \text{(\textit{Thus its perimeter is} } < 2\pi/\sqrt{k''} \text{ \textit{if} } k'' > 0.)$$

(iii) *With α'_{ij} (α''_{ij}) being the angle of that triangle opposite to the side of the length l_{ij}, each allowable matrix (α_{ij}) satisfying*

$$(2) \qquad \alpha'_{ij} \le \alpha_{ij} \le \alpha''_{ij}, \qquad i, j = 1, 2, \dots, m,$$

has a nondegenerate realization $a_1 a_2 \cdots a_m$ in S^{n-1}.

Then the matrix (l_{ij}) has a realization $p_0 p_1 \cdots p_m$ in M^n which fits the frame $\{e_1, e_2, \dots, e_m\}$. Moreover, any realization $p_0 p_1 \cdots p_k$, $k < m$, of the matrix (l_{ij}) with $i, j = 0, 1, \dots, k$ fitting the frame $\{e_1, e_2, \dots, e_k\}$ (we do not know if such a realization is unique) can be augmented by points $p_{k+1}, p_{k+2}, \dots, p_m$ such that the resulting set $p_1 p_2 \cdots p_m$ is a realization of the original matrix (l_{ij}) fitting the frame $\{e_1, e_2, \dots, e_m\}$.

2. Some related questions

Remark 1. Condition (ii) of the theorem is not easy to check directly. Theorem 2 in [3] yields a simple sufficient condition for (iii) to hold. Take some

$$(3) \qquad \alpha \ge \max_{i,j} \alpha''_{ij}$$

and suppose that

$$(4) \qquad \alpha \le 2 \arcsin \sqrt{\frac{m}{2(m-1)}}.$$

Then, by [3, Theorem 2], a quantity

$$(5) \qquad \lambda = \lambda(m-1, \alpha) \in (0, \alpha)$$

is determined identically by the equation

$$(6) \qquad \sin\lambda = \sin\alpha[1 - f(m-1)(\cos^2 R)/\cos^2(\alpha/2)]^{1/2}$$

where

$$(7) \qquad f(m-1) = \begin{cases} 2/m & \text{if } m \text{ is even}, \\ 2m/(m-1)(m+1) & \text{if } m \text{ is odd} \end{cases}$$

and

$$(8) \qquad \sin R = \sqrt{2(m-1)/m}\,\sin(\alpha/2).$$

(We use $m-1$ as the integer argument to comply with [3].) This λ has the property that each allowable $m \times m$ matrix (α_{ij}) with

$$(9) \qquad \alpha_{ij} \in (\lambda, \alpha], \qquad i \ne j,$$

is nondegenerate in S^{m-1}. Thus if $\min_{i,j} \alpha'_{ij} > \lambda = \lambda(m-1, \alpha)$ then (iii) holds.

Remark 2. A better though less convenient method to check condition (iii) arises from Theorem 1 in [3]. Put

$$s_{ij} = \cos\alpha_{ij} - \cos\alpha_{im}\cos\alpha_{jm}, \qquad i, j = 1, 2, \dots, m-1.$$

Theorem 1 in [3] implies that the $m \times m$ matrix (α_{ij}) of our theorem has a nondegenerate realization in S^{m-1} if and only if the $(m-1) \times (m-1)$ matrix $S = (s_{ij})$ has a positive spectrum. To establish this property of $S = S(\alpha_{ij})$ for each combination of the $m(m-1)/2$ arguments α_{ij} in the domain $D\colon a'_{ij} \leq \alpha_{ij} \leq \alpha''_{ij}$, it is enough to establish this property for one such combination and then make sure that $\det S(\alpha_{ij})$ stays positive in D.

Remark 3. Suppose the sectional curvature is constant in $N_r(p)$. Then one can take $k' = k'' = k$. The matrix (α_{ij}) is now unique. Its realizability can be checked with the help of Theorems 1 and 2 in [3]. Our theorem means now that each allowable matrix (l_{ij}) is freely realizable in $N_r(p)$ with 0th vertex at p (see the next remark) if and only if it satisfies (i), (ii), and (iii). (Considering necessity of (iii), one should notice that fitting the frame $\{e_1, e_2, \dots, e_m\}$ implies nondegeneracy of the realization $p_0 p_1 \cdots p_m$ in $N_r(p)$, which in turn implies nondegeneracy of the realization $a_1 \cdots a_m$ in S^{n-1}.) In this form, the theorem can be applied to spaces of constant curvature more general than X_k^n.

Let, for instance, $M^3 = S^1 \times R^2$. Then the convexity radius $R(p) = \pi/2$ for any $p \in M^3$. Take $r = \pi/2$. Consider the matrix

$$L = \begin{pmatrix} 0 & l_{01} & l_{02} \\ l_{10} & 0 & l_{12} \\ l_{20} & l_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1.58 & 1.58 \\ 1.58 & 0 & 3.15 \\ 1.58 & 3.15 & 0 \end{pmatrix}$$

where $1.58 > \pi/2$ and $3.15 > \pi$. This matrix has realizations in M^3, say, those located in R^2. The theorem, however, does not guarantee existence of any realization of L since $l_{01} > r$ in violation of (i). Importance of (i) becomes clear if one notices that L has no realization in $S^1 \times R^1 \subset M^3$ which would be symmetric about R^1. At the same time, replacing 1.58 and 3.15 by 1.57 and 3.13, one gets a matrix freely realizable in M^3 (see Remark 5 for an exact definition) according to the theorem.

Remark 4. Let K be a nonempty set in M^n and let (l_{ij}) be an allowable matrix. If for any $p \in K$ and any frame $\{e_1, e_2, \dots, e_m\}$, $2 \leq m \leq n$, at p the matrix (l_{ij}) has a realization in M^n fitting this frame, we will say that (l_{ij}) is *freely realizable in M^n with 0th vertex in K*. The theorem gives a simple sufficient condition of such realizability. Suppose that $\inf_{p\in K} R(p) > 0$ where $R(p)$ is the radius of convexity. Take a positive $r \leq \inf_{p\in K} R(p)$ and let k', k'' be finite lower and upper bounds of sectional curvature in the r-neighbourhood of K. Suppose that conditions (i), (ii), and (iii) hold with these r, k', k''. The theorem implies then that (l_{ij}) is freely realizable in M^n with 0th vertex in K.

Remark 5. Let P be a permutation on $\{0, 1, \dots, m\}$. Put $l_{ij}^P = l_{P(i)P(j)}$, $i, j = 0, 1, \dots, m$. If the matrix (l_{ij}^P) is freely realizable in M^n with 0th vertex in K for any P, we will say that the original matrix (l_{ij}) is *freely realizable in M^n with a vertex in K*. If $K = M^n$ here, we will say that (l_{ij}) is *freely realizable in M^n*. (In this case, M^n of course should be complete.)

Free realizability with 0th vertex in K does not imply free realizability with a vertex in K. In the rest of this remark, we assume that r, k', and k'' are as in the preceding remark. Note that conditions (i), (ii), and (iii) can hold for (l_{ij}) but fail for (l_{ij}^P). Of course, if (i), (ii), and (iii) hold for each matrix (l_{ij}^P) then (l_{ij}) is freely realizable with a vertex in K.

A natural question to ask in this connection is as follows. Suppose that (l_{ij}) is freely realizable in M^n with 0th vertex in K. Put

$$\tilde{l} = \max_i l_{0i};$$

$$\widetilde{K} = \begin{cases} \{x \in K | \rho(x, \partial K) \geq \tilde{l}\} & \text{if } \partial K \neq \varnothing, \\ M^n & \text{if } \partial K = \varnothing \text{ (then } K = M^n). \end{cases}$$

Suppose $\widetilde{K} \neq \varnothing$. It is guaranteed now that each realization $p_0 p_1 \cdots p_m$ of (l_{ij}) with at least one vertex, say p_1, in $\widetilde{K}$ has $p_0 \in K$. Let $F = \{f_1, f_2, \dots, f_m\}$ be a frame at the point $p_1 \in \widetilde{K}$. Is it possible to select $p_0 \in K$ and a frame $\{e_1, e_2, \dots, e_m\}$ at p_0 such that the realization $p_0 p_1 \cdots p_m$ would fit both frames? In other words, is (l_{ij}) freely realizable in M^n with a vertex in $\widetilde{K}$? We do not know the answer.

Remark 6. Let K, $r > 0$, k', and k'' be as in Remark 4. Let $L = (l_{ij})$, $i, j = 0, 1, \dots, m$, be the matrix of the edge lengths of a nondegenerate Euclidean m-simplex. Put $L(\varepsilon) = (\varepsilon l_{ij})$, $\varepsilon > 0$. Then $L(\varepsilon)$ is freely realizable in M^n with 0th vertex in K for sufficiently small ε. Indeed, together with the angles $\alpha'_{ij} = \alpha'_{ij}(\varepsilon)$ and $a''_{ij} = \alpha''_{ij}(\varepsilon)$ on k'- and k''-planes for the matrix $L(\varepsilon)$, consider also the appropriate angles α^0_{ij} on Euclidean plane. (They do not depend on ε.) The matrix (α^0_{ij}) is nondegenerate in S^{m-1} since L is nondegenerate in R^m. Obviously, $\alpha'_{ij} \to \alpha^0_{ij}$, $\alpha''_{ij} \to \alpha^0_{ij}$ as $\varepsilon \to 0$. Then, for sufficiently small ε, any matrix (α_{ij}) with $\alpha_{ij} \in [\alpha'_{ij}(\varepsilon), \alpha''_{ij}(\varepsilon)]$ is arbitrarily close to the nondegenerate (α^0_{ij}). Since nondegenerate matrices form an open set in the appropriate matrix space (see [3, Corollary of Theorem 1]), these matrices (α_{ij}) are nondegenerate. Now the theorem implies that $L(\varepsilon)$ is freely realizable in M^n with 0th vertex in K.

Applying this observation to each permuted matrix l_{ij}^P (see Remark 5), one will see also that $L(\varepsilon)$ is freely realizable in M^n with a vertex in K when ε is sufficiently small.

Remark 7. Note finally that the theorem does not assume triangle inequalities involving l_{ij}, l_{ik}, l_{jk} in which the index 0 does not appear among i, j, k. Those triangle inequalities follow from the theorem, i.e., from realizability of the matrix (l_{ij}).

3. Proof of the theorem

A. One may consider only the case $k = m - 1$ since points can be added one at a time. We use induction by m. If $m = 2$, one obviously can select p_0 and p_1 fitting the frame $\{e_1\}$. (This selection is unique in this particular case.) Let $p(\phi)$ be such that the length $p_0 p(\phi) = l_{02}$, and the direction of the segment $p_0 p(\phi)$ is coplanar with e_1 and e_2 and forms an angle $\phi \in [0, \pi]$ with e_1 and an angle $\leq \pi/2$ with e_2. The distance $p_1 p(\phi)$ changes continuously with ϕ

from $p_1p(0) = |l_{01} - l_{02}| \le l_{12}$ to $p_1p(\pi) = l_{01} + l_{02} \ge l_{12}$. Then $p_1p(\phi^*) = l_{12}$ for some $\phi^* \in [0, \pi]$. If $\phi^* = \pi$ then $l_{12} = l_{01} + l_{02}$, which is impossible for the nondegenerate triangles mentioned in (ii). Thus $\phi^* \ne \pi$. If $\phi^* = 0$, then either $l_{01} - l_{02} = l_{12}$ or $l_{02} - l_{01} = l_{12}$, which is also impossible due to (ii). Hence $\phi^* \in (0, \pi)$ and the point $p_2 = p(\phi^*)$ is a desirable one. (We do not know if ϕ^* and p_2 are unique.) Thus the theorem, including the statement on augmentation, holds for $m = 2$.

B. Suppose now that the theorem holds for $m - 1 \ge 2$ in place of m. Along with the matrix $L = (l_{ij})$, $i, j = 0, 1, \dots, m$, we will consider three other matrices: L_m with is obtained from L by deleting its mth, i.e., the last, row and column; L_{m-1} obtained from L by deleting its $(m-1)$st row and column; and L_{m-1m} obtained from L by deleting its last two rows and last two columns. Similarly, we introduce three modifications, A_m, A_{m-1}, and A_{m-1m}, of a matrix $A = (\alpha_{ij})$, $i, j = 1, 2, \dots, m$. Note that since A has a nondegenerate realization in S^{n-1} for any choice of its elements $\alpha_{ij} \in [\alpha'_{ij}, \alpha''_{ij}]$, the same is true of A_m, A_{m-1}, and A_{m-1m}. By our induction assumption, there exists a realization $p_0p_1 \cdots p_{m-1}$ of L_m fitting the frame $\{e_1, e_2, \dots, e_{m-1}\}$. For $\phi \in [0, \pi]$, denote by $e(\phi)$ the unit vector coplanar to e_{m-1} and e_m forming an angle ϕ with e_{m-1} and an angle $\le \pi/2$ with e_m. Obviously the part $p_0p_1 \cdots p_{m-2}$ of the last realization is a realization of L_{m-1m}. By our induction assumption, this realization $p_0p_1 \cdots p_{m-2}$ can be augmented by a point $p(\phi)$ such that the resulting set $p_0p_1 \cdots p_{m-2}p(\phi)$ is a realization of L_{m-1} fitting the frame $\{e_1, e_2, \dots, e_{m-2}, e(\phi)\}$. Denote by $a_1, \dots, a_{m-1}, a(\phi)$ the directions of the segments $p_0p_1, \dots, p_0p_{m-1}, p_0p(\phi)$ at p_0. We now specify the entries a_{ij} of the matrix A above as follows. We assume a_{ij} to be the distance a_ia_j on the sphere S^{n-1} of directions at p_0 for $i, j \le m - 1$, i.e.,

$$\alpha_{ij} = a_ia_j = \angle p_ip_0p_j, \qquad i, j = 1, 2, \dots, m-1. \tag{10}$$

We put

$$\alpha_{im} = \alpha_{mi} = a_ia(\phi), \quad i = 1, 2, \dots, m-1, \qquad \alpha_{mm} = 0. \tag{11}$$

Thus $a_1 \cdots a_{m-1}a(\phi)$ is now a realization of the matrix A in S^{n-1} while $a_1 \cdots a_{m-1}$, $a_1 \cdots a_{m-2}a(\phi)$, and $a_1 \cdots a_{m-2}$ are realizations of A_m, A_{m-1}, and A_{m-1m}. Note that in case $M^n = X^n_k$, the entries α_{im} do not depend on ϕ except for α_{m-1m}.

C. We make now an important reference to comparison theorems for triangles by Alexandrow and Toponogov. That will be the only substantial reference to Riemannian Geometry in this paper. Since $l_{0i} < r$, all our points $p_0, p_1, \dots, p_{m-1}, p(\phi)$ and the segments between them lie in $N_r(p_0)$. According to [2, §6.4.2, Theorem and Remark 3], Toponogov's Theorem can be stated as follows.

Theorem (V. A. Toponogov). *Let C be a convex set in M^n, $n \ge 2$, and k' be a lower bound of the sectional curvature at points of C. Then for any triangle made of shortest paths in C there exists a triangle in the k'-plane with the same side lengths such that the angles α, β, γ of the triangle in C and the corresponding angles α', β', γ' of the triangle in the k'-plane satisfy*

$$\alpha' \le \alpha, \qquad \beta' \le \beta, \qquad \gamma' \le \gamma. \tag{12}$$

Since $N_r(p_0)$ is convex, the theorem applies to it and yields

$$\alpha'_{ij} \leq \alpha_{ij}, \qquad i, j = 1, 2, \dots, m-1; \tag{13}$$

$$\alpha'_{im} \leq \alpha_{im} = a_i a(\phi) = \angle p_i p_0 p(\phi) \quad \text{for } i \leq m-2 \text{ and } \phi \in [0, \pi]. \tag{14}$$

The local comparisons of the angles of triangles like those in (13) and (14) were actually understood prior to Toponogov's global results, e.g., by Alexandrow.

Note that $\alpha_{m-1m} = \angle p_{m-1} p_0 p(\phi)$ is not involved in either (13) or (14) since, generally speaking, $p_{m-1}p(\phi) \neq l_{m-1m}$. (We are just working towards the appropriate equality.) Denote by B the closed metric ball centered at p_0 whose radius is $\max_{1 \leq i \leq m} l_{i0}$. Since this radius is $< r$, the ball B is convex. According to [1, §1.7b)], B is a domain of type $R_{k''}$ (defined in [1, §1.4]). It follows from [1, the end of §1.6 and §1.4c)] that, for any triangle in B of perimeter $< 2\pi\sqrt{k''}$, if $k'' > 0$, the triangle in the k''-plane with the same side lengths satisfies

$$\alpha \leq \alpha'', \qquad \beta \leq \beta'', \qquad \gamma \leq \gamma'' \tag{15}$$

where α, β, γ are the angles of the triangle in B and $\alpha'', \beta'', \gamma''$ are the corresponding angles of the triangle on the k''-plane. Due to (1), the estimate (15) can be applied to the triangles $p_i p_0 p_j$ and $p_i p_0 p(\phi)$ resulting in

$$\alpha_{ij} \leq \alpha''_{ij}, \qquad i, j = 1, 2, \dots, m-1; \tag{16}$$

$$\alpha_{im} = \angle p_i p_0 p(\phi) \leq \alpha''_{im} \quad \text{for } i \leq m-2 \text{ and } \phi \in [0, \pi]. \tag{17}$$

The relations (13), (14), (16), and (17) mean that the off-diagonal elements of the matrices A_m and A_{m-1} satisfy condition (2). Therefore their realizations $a_1 \cdots a_{m-1}$ and $a_1 \cdots a_{m-2} a(\phi)$ are nondegenerate.

D. It will be convenient to associate with these two realizations the nondegenerate spherical $(m-2)$-simplexes $a_1 a_2 \cdots a_{m-1}$ and $a_1 a_2 \cdots a_{m-2} a(\phi)$ of which the first one is immovable while the second one varies with ϕ. The variation, however, is not rotation about the common $(m-3)$-face $a_1 a_2 \cdots a_{m-2}$ since the lengths of the edges $a_i a(\phi)$, $i \leq m-2$, generally speaking depend on ϕ (see (11)) unless $M^n = X^n_k$. In subsections E, F, G, and H we are going to watch the distance $\alpha(\phi) = a_{m-1} a(\phi)$ in S^{n-1} as ϕ varies on $[0, \pi]$.

E. Let $S^{m-2} \subset S^{n-1}$ be the sphere determined by $a_1 a_2 \cdots a_{m-1}$. Denote by H_0 and H_π the two closed semispheres of S^{m-2} whose common boundary is the spere S^{m-3} determined by $a_1 a_2 \cdots a_{m-2}$; see Figure 1 on the next page. Since the simplex $a_1 a_2 \cdots a_{m-1}$ is nondegenerate, the point $a_{m-1} \notin S^{m-3}$. One may assume that

$$a_{m-1} \in \operatorname{relint} H_0 = H_0 \backslash S^{m-3}. \tag{18}$$

Put

$$F_0 = \{x \in H_0 | x a_i \in [\alpha'_{im}, \alpha''_{im}], \ i = 1, 2, \dots, m-2\}; \tag{19}$$

$$F_\pi = \{x \in H_\pi | x a_i \in [\alpha'_{im}, \alpha''_{im}], \ i = 1, 2, \dots, m-2\}. \tag{20}$$

Let us show that

$$a(0) \in F_0, \qquad a(\pi) \in F_\pi. \tag{21}$$

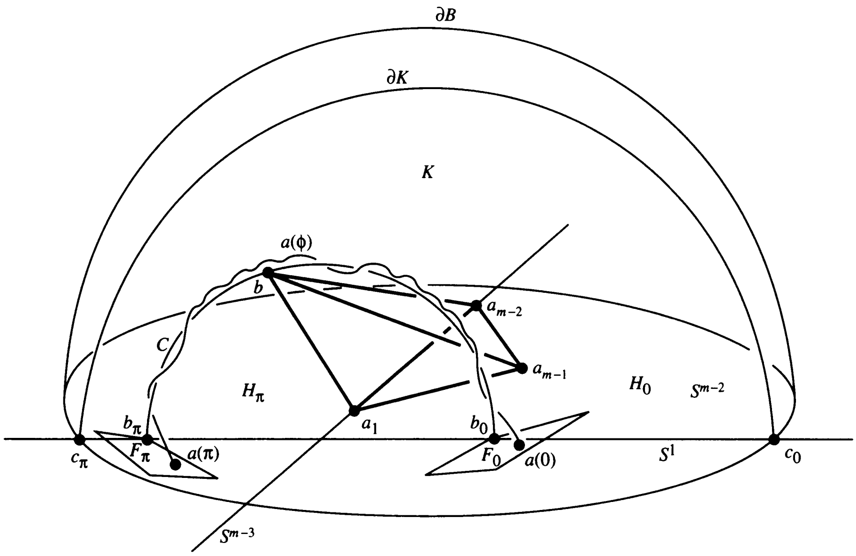

FIGURE 1

Since $p_0p_1\cdots p_{m-1}$ and $p_0p_1\cdots p_{m-2}p(0)$ both fit the same frame $\{e_1, e_2, \dots, e_{m-1} = e(0)\}$, the direction $a(0)$ of the segment $p_0p(0)$ lies in H_0 as the point a_{m-1} does according to (18). The distances $a_ia(0) = \alpha_{im}(0)$, $i \le m-2$ (see (11)), satisfy (2) according to (14) and (17). Thus $a(0) \in F_0$. Similarly $a(\pi) \in F_\pi$.

F. Denote by G the sperical shell

$$G = \{x \in S^{n-1} | \alpha'_{m-1m} \le xa_{m-1} \le \alpha''_{m-1m}\}. \tag{22}$$

Let us show that

$$F_0 \cap G = \varnothing; \qquad F_\pi \cap G = \varnothing. \tag{23}$$

Suppose to the contrary that, say, $F_0 \cap G \ni z$. Then the mutual distances of the m points $a_1, \dots, a_{m-1}, z$ in S^{n-1} satisfy (2), and hence $a_1 \cdots a_{m-1}z$ should be a nondegenerate $(m-1)$-simplex. On the other hand, these m points lie in S^{m-2}.

G. We prove now that the spherical shell

$$G \text{ separates } F_0 \text{ and } F_\pi. \tag{24}$$

Suppose to the contrary that, for instance,

$$F_0 \cup F_\pi \subset B \overset{\text{def}}{=} \{x \in S^{n-1} | xa_{m-1} < \alpha'_{m-1m}\}. \tag{25}$$

Consider the matrix

$$A' \overset{\text{def}}{=} \begin{pmatrix} 0 & \alpha_{12} & \cdots & \alpha_{1m-1} & \alpha'_{1m} \\ a_{21} & 0 & \cdots & \alpha_{2m-1} & \alpha'_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{m-11} & \alpha_{m-12} & \cdots & 0 & \alpha'_{m-1m} \\ \alpha'_{m1} & \alpha'_{m2} & \cdots & \alpha'_{mm-1} & 0 \end{pmatrix} \tag{26}$$

where α_{ij} are defined by (10). Due to (13) and (16), condition (2) holds for A'. Then A' has a nondegenerate realization. One obviously may assume that it lies in the $(m-1)$-sphere $S^{m-1} \subset S^{n-1}$ determined by the points $e_1, \dots, e_m$ and that the first $m-1$ points of the realization are $a_1, a_2, \dots, a_{m-1}$. Denote by b the last point of this realization. Then

$$b \in \partial B. \tag{27}$$

As the point b rotates in S^{m-1} about S^{m-3} (determined by $a_1 \cdots a_{m-2}$), it travels a circumference C (nondegenerate since $b \notin S^{m-3}$), see Figure 1. The 2-sphere S^2 determined by C is orthogonal to S^{m-3} and thus to S^{m-2}. Therefore the great circle $S^1 = S^2 \cap S^{m-2}$ includes a diameter $c_0 c_\pi$ of the circle $K = S^2 \cap B$, see Figure 1. (By a diameter, we mean here a longest geodesic in K that can be longer than π when the radius α'_{m-1m} of B is $> \pi/2$.) Due to (27),

$$b \in C \cap \partial K. \tag{28}$$

Note that the points b_0 and b_π which make up $C \cap S^1 = C \cap S^{m-2}$ satisfy

$$a_i b_0 = a_i b_\pi = \alpha'_{im}, \qquad i = 1, 2, \dots, m-2. \tag{29}$$

Thus each of them is either in F_0 or F_π; however, if one is in F_0 then the other one should be in F_π since b_0 and b_π are distinct and symmetric about S^{m-3}. One may assume that

$$b_0 \in F_0; \qquad b_\pi \in F_\pi. \tag{30}$$

By contrary assumption (25), b_0 and b_π lie in the open ball B. Therefore b_0 and b_π are interior points of the diameter $c_0 c_\pi$, see Figure 1. Obviously the circumference C is orthogonal to $c_0 c_\pi$ at b_0 and b_π. Then C and ∂K cannot intersect contrary to (28). The case $F_0 \cup F_\pi \subset S^{m-1} \backslash (B \cup G)$ reduces to a contradiction, similarly, which proves (24).

H. Since the points $b_0, b_\pi \in S^{m-2}$ are symmetric about S^{m-3} and (see (30)) $b_0 \in F_0$ thus lying on the same side of S^{m-3} with a_{m-1}, the distance

$$b_0 a_{m-1} \le b_\pi a_{m-1}. \tag{31}$$

Now if $F_\pi \subset B$ and, by (24), $F_0 \subset S^{m-1} \backslash (G \cup B)$ then by (30) the distance $b_\pi a_{m-1} < b_0 a_{m-1}$ contrary to (31). Hence

$$F_0 \subset B, \qquad F_\pi \subset S^{m-1} \backslash (G \cup B). \tag{32}$$

Now come back to the distance $\alpha(\phi) = a_{m-1} a(\phi)$ singled out in subsection D. Due to (32) and (21), one has

$$\alpha(0) < \alpha'_{m-1m} \le \alpha''_{m-1m} < \alpha(\pi). \tag{33}$$

I. Now we watch the distance $p_{m-1} p(\phi)$ in M^n. For the triangle $p_{m-1} p_0 p(\phi)$, consider in the k'-plane (k''-plane) a triangle with the same side lengths. Denote by $\alpha'(\phi)$ ($\alpha''(\phi)$) its angle opposite to the side of the length $p_{m-1} p(\phi)$. Due to (12) and (15),

$$\alpha'(\phi) \le \alpha(\phi) \le \alpha''(\phi). \tag{34}$$

Suppose now that $p_{m-1} p(0) \ge l_{m-1m}$. Due to (34) and geometry of the k-plane, one has then $\alpha(0) \ge \alpha'(0) \ge \alpha'_{m-1m}$ contrary to (33). Thus $p_{m-1} p(0) < l_{m-1m}$.

Similarly, $p_{m-1}p(\pi) > l_{m-1m}$. By continuity, $p_{m-1}p(\phi^*) = l_{m-1m}$ at some $\phi^* \in (0, \pi)$.

Thus an arbitrary realization $p_0p_1 \cdots p_{m-1}$ of L_m fitting the frame $\{e_1, e_2, \dots, e_{m-1}\}$ has been augmented by the point $p_m = p(\phi^*)$ such that $p_0p_1 \cdots p_m$ is a realization of L. Obviously this realization fits the frame $\{e_1, e_2, \dots, e_m\}$. This completes the proof. (Again, we do not know if ϕ^* and p_m are unique.)

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