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A NOTE ON WEIGHTED SOBOLEV SPACES, AND REGULARITY OF COMMUTATORS AND LAYER POTENTIALS ASSOCIATED TO THE HEAT EQUATION

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ABSTRACT. We give a simplified proof of recent regularity results of Lewis and Murray, namely, that certain commutators, and the boundary single layer potential for the heat equation in domains in $\mathbb{R}^2$ with time dependent boundary, map L^p into an appropriate homogeneous Sobolev space. The simplification is achieved by treating directly only the case $p = 2$, but in a weighted setting.

1. INTRODUCTION AND STATEMENT OF RESULTS

Regularity results for certain commutators and layer potentials associated to the heat equation in domains in $\mathbb{R}^2$ with time dependent boundary have recently been obtained by Lewis and Murray [LM]. They showed that these operators are bounded from L^p into an appropriate Sobolev space $I_\alpha(L^p)$. Their proof proceeds in two steps: First treat the case $p = 2$, and then use a variety of real variable techniques to extend to the case $p \neq 2$. Their results are equivalent to the L^p boundedness of certain "nonstandard" singular integrals that, in particular, need not map constants into BMO; thus the T1 Theorem does not apply, nor can one interpolate with an end point estimate to obtain the case $2 < p < \infty$. Not surprisingly then, the second part of their program entails a not inconsiderable expenditure of effort, and it would therefore seem desirable to dispense with this step entirely. Fortunately, there is a way to do this: in the (possibly appocryphal) words of Rubio de Francia, "L^p does not exist, only (weighted)L^2." In this note we will prove a weighted version of the L^2 result of [LM], from which most of the L^p theory follows automatically (in particular, we obtain the case of principal interest in parabolic theory, namely, $\alpha = \frac{1}{2}$ for all p, $1 < p < \infty$). While the weighted results are new and perhaps of independent interest, our primary motivation in establishing them is to simplify the arguments of [LM].

Before stating our theorems, we need to recall some elementary facts about Littlewood-Paley theory in $\mathbb{R}^n$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be radial, be supported in the unit ball, and have mean value zero. We define $Q_s f \equiv \psi_s * f$, where $\psi_s(x) \equiv$

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$s^{-n}\psi(x/s)$, and where ψ has been normalized so that $\int_0^\infty(\hat{\psi}(s\xi))^2\,ds/s = 1$ for all $\xi \in \mathbb{R}^n$ (this can be done since ψ is radial). Thus Q_s satisfies the "Calderon reproducing formula"

$$\int_0^\infty Q_s^2 \frac{ds}{s} = I.$$

For $0 < \alpha < 1$, we define $\widetilde{Q}_s \equiv s^{-\alpha} I_\alpha Q_s$, where as usual I_α denotes the fractional integral operator

$$(I_\alpha f)\hat{\ }(\xi) \equiv |\xi|^{-\alpha}\hat{f}(\xi).$$

Then, at least for test functions,

$$\int_0^\infty \widetilde{Q}_s^2 \frac{ds}{s} = C_\alpha I$$

since

$$0 < \int_0^\infty s^{-2\alpha}(\hat{\psi}(s))^2 \frac{ds}{s} = C_\alpha < \infty, \qquad 0 < \alpha < 1.$$

The latter inequality follows from the smoothness of $\hat{\psi}$ and the fact that $\hat{\psi}(0) = 0$ (so, in particular, $|\hat{\psi}(s)/s|$ is bounded near the origin). If we set

$$\hat{\tilde{\psi}}(|\xi|) \equiv |\xi|^{-\alpha}\hat{\psi}(|\xi|)$$

then a routine computation shows that

$$|\tilde{\psi}_s(x)| \leq \frac{C_{n,\alpha}s^{1-\alpha}}{(s+|x|)^{n+1-\alpha}} \quad \text{and} \quad |\nabla\tilde{\psi}_s(x)| \leq \frac{C_{n,\alpha}s^{1-\alpha}}{(s+|x|)^{n+2-\alpha}}.$$

Thus,

$$\left(\int_0^\infty |\tilde{\psi}_s(x)|^2 \frac{ds}{s}\right)^{1/2} \leq C|x|^{-n}$$

and

$$\left(\int_0^\infty |\nabla\tilde{\psi}_s(x)|^2 \frac{ds}{s}\right)^{1/2} \leq C|x|^{-n-1}.$$

By vector-valued Calderon-Zygmund theory (see, e.g., [GR, Chapter V]), we then have, for all $w \in A_2$,

$$\|\tilde{g}(f)\|_{2,w} \approx \|f\|_{2,w}, \tag{1.1}$$

where

$$\tilde{g}(f) \equiv \left(\int_0^\infty |\widetilde{Q}_s f|^2 \frac{ds}{s}\right)^{1/2}.$$

Furthermore, the homogeneous weighted Sobolev space $I_\alpha(L^2_w)$, $w \in A_2$, can be given the norm

$$\|f\|_{I_\alpha(L^2_w)} \equiv \left(\int_{\mathbb{R}^n}\int_0^\infty |Q_s f(x)|^2 \frac{ds}{s^{1+2\alpha}} w(x)\,dx\right)^{1/2}, \tag{1.2}$$

because by (1.1) this last expression is comparable to $\|\tilde{f}\|_{2,w}$, where $f = I_\alpha(\tilde{f})$.

For $0 < \alpha < 1$, let (real-valued) $A \in I_\alpha(\mathrm{BMO})$, and consider the one-dimensional operator

$$K_\alpha f(x) \equiv \int_{\mathbb{R}} \frac{[A(x)-A(y)]^2}{|x-y|^{1+\alpha}} f(y)\,dy. \tag{1.3}$$

As in [LM], our results for this operator can easily be extended to the boundary single layer potential for the heat equation in domains $\{(x_1, x_2)\colon x_1 > A(x_2)\}$, with $A = I_{1/2}a$, $a \in \mathrm{BMO}$. We shall return to this point in §4. Our principal result is the following.

Theorem 1.4. *Let* $A = I_\alpha a$, *and suppose* $w \in A_1$ *if* $0 < a < 1$ *or* $w \in A_2$ *if* $\frac{1}{2} \le \alpha < 1$. *Then*

$$\|K_\alpha f\|_{I_\alpha(L^2_w)} \le C_\alpha \|a\|_*^2 \|f\|_{2,w}.$$

As an almost immediate corollary, we recover, except for the case $1 < p < 2$, $0 < \alpha < \frac{1}{2}$, the result of [LM, Theorem 3].

Theorem 1.5. *Suppose* $1 < p < \infty$ *if* $\frac{1}{2} \le \alpha < 1$, *or* $2 \le p < \infty$ *if* $0 < \alpha < 1$. *Then*

$$\|K_\alpha f\|_{I_\alpha(L^p)} \le C_{\alpha,p} \|a\|_*^2 \|f\|_p.$$

Proof of Theorem 1.5 (Modulo Theorem 1.4). If $\frac{1}{2} \le \alpha < 1$, then by Theorem 1.4 we have that $D^\alpha K_\alpha$ is bounded on L^2_w, $w \in A_2$, and therefore on L^p_w, $1 < p < \infty$, $w \in A_p$, by Rubio de Francia's extrapolation theorem (see, e.g., [GR, Chapter IV]), where

$$(D^\alpha f)\hat{}\,(\xi) \equiv |\xi|^\alpha \hat{f}(\xi).$$

If $0 < \alpha < 1$ then $D^\alpha K_\alpha$ is bounded on L^2_w, $w \in A_1$. In particular, by a result of Coifman and Rochberg [CR], we have for $u \in L^{(p/2)'}$, $p > 2$, the inequality

$$\int |D^\alpha K_\alpha f(x)|^2 u(x)\,dx \le C_{\alpha,\varepsilon} \|a\|_*^4 \int |f(x)|^2 (M(|u|^{1+\varepsilon}))^{1/1+\varepsilon}(x)\,dx,$$

for any positive ε. By choosing $1+\varepsilon < (p/2)'$, the L^p boundedness, $p > 2$, of $D^\alpha K_\alpha$ may be deduced by a standard duality argument. Theorem 1.5 follows.

We remark that the proof to follow will actually show that in the case $0 < \alpha < \frac{1}{2}$, one may take $w \in A_{1+2\alpha}$ in Theorem 1.4, and therefore by a slightly more involved duality argument, one obtains Theorem 1.5 for $p > 2/(1+2\alpha)$. This is the best result that can be directly obtained by our method, which relies on the auxilary use of the Littlewood-Paley g_λ^* function, with $\lambda < 1 + 2\alpha$. Since the full range of p has already been treated in [LM], we shall content ourselves with Theorems 1.4 and 1.5 as stated.

In the next section, we give a transparent extension to the weighted setting of a result of Strichartz relating Carleson measures and $I_\alpha(\mathrm{BMO})$. In §3 we prove Theorem 1.4, and then in §4 we describe how this result may be extended to the boundary single layer potential.

2. I_α(BMO) AND WEIGHTED CARLESON MEASURES

We first need a preliminary fact.

Lemma 2.1. *With* $0 < \alpha < 1$, *the square function* $g_\alpha f$ *defined on* $\mathbb{R}^n$ *by*

$$g_\alpha f(x) \equiv \left(\int_{\mathbb{R}^n} |I_\alpha f(x+h) - I_\alpha f(x)|^2 \frac{dh}{|h|^{n+2\alpha}}\right)^{1/2}$$

is bounded on L^2_w, $w \in A_{p(\alpha)}$, *where* $p(\alpha) \equiv \min(1 + 2\alpha/n, 2)$. *In particular, we may always take* $w \in A_1$, *and if* $n = 1$ *and* $\frac{1}{2} \le \alpha < 1$, *we may take* $w \in A_2$.

Proof of Lemma 2.1. This is fairly trivial. First (see, e.g., Stein [S, pp. 162–163, 6.12, 6.13] and the references given therein) we have the pointwise bound

$$g_\alpha f(x) \le C_{\alpha,\lambda} g^*_\lambda f(x) \tag{2.2}$$

if $\lambda < 1 + 2\alpha/n$ (see [S, p. 88] for the definition of g^*_λ). But by a result of Muckenhoupt and Wheeden [MW], g^*_λ is bounded on L^2_w, with $w \in A_{q(\lambda)}$, $q(\lambda) \equiv \min(\lambda, 2)$, $\lambda > 1$. If $1 + 2\alpha/n > 2$, we take $\lambda = 2$. If $1 + 2\alpha/n \le 2$ and $w \in A_{1+2\alpha/n}$, then by a well-known property of A_p weights we may select a $\lambda < 1 + 2\alpha/n$, with $w \in A_\lambda$. In either case, Lemma 2.1 follows by [MW] and the pointwise bound (2.2).

We now give a weighted version of Theorem (3.3) of [Stz].

Lemma 2.3. *Suppose* $0 < \alpha < 1$ *and* $A = I_\alpha a$, $a \in BMO$. *If* $Q(s)$ *is a cube with side length* s *and* $p(\alpha) \equiv \min(1 + 2\alpha/n, 2)$, *then*

$$\frac{1}{w(Q(s))} \int_{Q(s)} \int_{|h| \le s} \frac{|A(x+h) - A(x)|^2}{|h|^{n+2\alpha}} \, dh w(x)\, dx \le C_\alpha \|a\|_*^2, \tag{2.4}$$

where $w \in A_{p(\alpha)}$. *In particular, we may always take* $w \in A_1$, *and if* $\frac{1}{2} \le \alpha < 1$ *and* $n = 1$, *we may take* $w \in A_2$.

Proof of Lemma 2.3. This is easy if we follow the argument in [Stz], combined with that of Journe [J, pp. 85–87], so we only give a brief sketch. Since the operator

$$f \to I_\alpha f(x+h) - I_\alpha f(x) \tag{2.5}$$

has Fourier multiplier $[e^{ih\cdot\xi} - 1]|\xi|^{-\alpha}$, it annihilates constants, so we may assume that a has mean value zero on $Q^*(s)$. Here $Q^*(s)$ denotes the cube concentric with $Q(s)$ and has ten times the diameter of $Q(s)$. As usual, we write $a = a_1 + a_2$, where $a_1 \equiv a\chi_{Q^*(s)}$, $a_2 = a\chi_{(Q^*(s))^C}$. Now crudely, by Lemma 2.1, the left side of (2.4) with $I_\alpha a_1$ in place of A is no larger than a constant times

$$\frac{1}{w(Q(s))} \int |a_1(x)|^2 w(x)\, dx.$$

The desired estimate for this last term may be obtained exactly like the corresponding estimate in [J, p. 86] by using Hölder's inequality, the reverse Hölder property of A_p weights, the John-Nirenberg Theorem, and the fact that w defines a doubling measure. To handle the part of (2.4) corresponding to $I_\alpha a_2$, we observe first that the operator defined by (2.5) is given by convolution with the kernel

$$C_n\left[\frac{1}{|x+h|^{n-\alpha}}-\frac{1}{|x|^{n-\alpha}}\right]\leq C_{n,\alpha}\frac{|h|}{|x|^{n+1-\alpha}},$$

where the last inequality holds whenever $|x|>2|h|$. If we write $h=t\theta$ in polar coordinates, then the left side of (2.4) with $I_\alpha a_2$ in place of A is bounded by

$$\int_{S^{n-1}}\int_{Q(s)}\int_0^s\left[\int_{\mathbb{R}^n}\frac{t^{1-\alpha}}{(t+|x-y|)^{n+1-\alpha}}|a_2(y)|\,dy\right]^2\frac{dt}{t}w(x)\,dx\,d\theta. \tag{2.6}$$

For $x\in Q(s)$ and $y\in(Q^*(s)^c$, we have $(s+|x-y|)\approx|x-y|\approx(t+|x-y|)$. Thus, the expression in square brackets in (2.6) is dominated by a constant times

$$\left(\frac{t}{s}\right)^{1-\alpha}\int_{\mathbb{R}^n}\frac{s^{1-\alpha}}{(s+|x-y|)^{n+1-\alpha}}|a_2(y)|\,dy\leq C\left(\frac{t}{s}\right)^{1-\alpha}\|a\|_*,$$

where the last inequality follows by a slight variant of a classical argument of Fefferman Stein [FS] (see, e.g., [Stz, Lemma 2.2]). Lemma 2.3 may now be obtained by plugging this last expression into (2.6).

3. Proof of Theorem 1.4

The proof is based on ideas developed by Lewis and Murray in [LM, §3]. Our objective is to prove

$$\int_{\mathbb{R}}\int_0^\infty|Q_sK_\alpha f(x)|^2\frac{ds}{s^{1+2\alpha}}w(x)\,dx\leq C_\alpha\int_{\mathbb{R}}|f(x)|^2w(x)\,dx, \tag{3.1}$$

where, without loss of generality, we assumed that $\|a\|_*=1$ (recall that $A=I_\alpha a$, $a\in\mathrm{BMO}$). Here, $w\in A_2$ if $\frac{1}{2}\leq\alpha<1$, or $w\in A_1$ if $0<\alpha<1$. We smoothly truncate the kernel of K_α as follows. Choose a radial $\varphi\in C_0^\infty$, $0\leq\varphi\leq 1$, where $\varphi\equiv 1$ on $\{|x|<100\}$ and $\varphi\equiv 0$ on $\{|x|>101\}$. For fixed s, we write

$$\begin{aligned}\frac{[A(x)-A(y)]^2}{|x-y|^{1+\alpha}}&\equiv[A(x)-A(y)]^2\left\{|x-y|^{-1-\alpha}\varphi\left(\frac{|x-y|}{s}\right)\right.\\&\qquad\left.+|x-y|^{-1-\alpha}\left(1-\varphi\left(\frac{|x-y|}{s}\right)\right)\right\}\\&\equiv[A(x)-A(y)]^2\{j_s(x-y)+k_s(x-y)\}.\end{aligned}\tag{3.2}$$

We consider first the term corresponding to j_s, which is essentially the same as θ_1 in [LM, (3.10)] (the term θ_2 in [LM] will not arise in the present argument and their term θ_3 corresponds to k_s). The part of the left side of (3.1) corresponding to j_s is crudely bounded by

$$\int_0^\infty\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|\psi_s(x-z)|(A(z)-A(y))^2j_s(z-y)|f(y)|\,dz\,dy\right]^2w(x)\,dx\,\frac{ds}{s^{1+2\alpha}}.$$

Since convolution with $|\psi_s|$ is controlled by the Maximal Function, by Muckenhoupt's theorem the last expression is no larger than a constant times

$$\int_0^\infty\int_{\mathbb{R}}\left[\int_{\mathbb{R}}(A(x)-A(y))^2j_s(x-y)|f(y)|\,dy\right]^2w(x)\,dx\,\frac{ds}{s^{1+2\alpha}}$$

for all $w \in A_2$. In analogy with [LM, (3.15)–(3.17)], we apply Minkowski's integral inequality to obtain the bound

$$\int_{\mathbb{R}} \left(\int \frac{A(x)-A(y)^2}{|x-y|^{1+\alpha}} |f(y)| \left(\int_{|x-y|/101}^{\infty} \frac{ds}{s^{1+2\alpha}} \right)^{1/2} dy \right)^2 w(x)\,dx$$
$$= C_\alpha \|C_2(|f|)\|_{2,w}^2 ,$$

where C_2 is the second-order fractional commutator with kernel

$$k^{(2)}(x,y) \equiv \frac{(A(x)-A(y))^2}{|x-y|^{1+2\alpha}}. \tag{3.3}$$

But C_2 is bounded on unweighted L^2 by a result of Murray [M2]. Furthermore, since $A \in I_\alpha(\mathrm{BMO}) \subseteq \mathrm{Lip}\,\alpha$, the kernel $k^{(2)}$ satisfies "standard" Calderon-Zygmund estimates, so C_2 is bounded on L_w^2, $w \in A_2$, by the usual arguments (see, e.g., Coifman and Fefferman [CF]).

We now turn to the part of (3.1) corresponding to k_s. We consider the kernel of the composition of Q_s with the operator

$$f \to \int [A(x)-A(y)]^2 k_s(x-y) f(y)\,dy.$$

Since $Q_s 1 = 0$, this kernel equals

$$\begin{aligned} &\int_{\mathbb{R}} \psi_s(x-z)[A(z)-A(y)]^2[k_s(z-y)-k_s(x-y)]\,dz \\ &\qquad + \int_{\mathbb{R}} \psi_s(x-z)([A(z)-A(y)]^2-[A(x)-A(y)]^2)k_s(x-y)\,dz \\ &\equiv H_s(x,y)+L_s(x,y). \end{aligned} \tag{3.4}$$

The terms H_s and L_s correspond to σ_1 and σ_2 in [LM, (3.24) and (3.25)]. We treat L_s first, and following [LM, (3.32)] we write

$$\begin{aligned} &(A(z)-A(y))^2-(A(x)-A(y))^2 \\ &\qquad = (A(z)-A(x))^2 + 2(A(z)-A(x))(A(x)-A(y)). \end{aligned} \tag{3.5}$$

Since $\int \psi = 0$, the part of L_s containing the second part of (3.5) equals twice

$$Q_s A(x)[A(x)-A(y)]k_s(x-y). \tag{3.6}$$

Recall that $A = I_\alpha a$, with $a \in \mathrm{BMO}$. Plugging (3.6) into (3.1) in place of $Q_s K_\alpha$, we obtain

$$\int_{\mathbb{R}} \int_0^\infty |\widetilde{Q}_s a(x) C_{1,100s} f(x)|^2 \frac{ds}{s} w(x)\,dx, \tag{3.7}$$

where $\widetilde{Q}_s \equiv s^{-\alpha} Q_s I_\alpha$ and $C_{1,100s}$ is the smoothly truncated first fractional commutator with kernel

$$\frac{A(x)-A(y)}{|x-y|^{1+\alpha}} \left(1-\varphi\left(\frac{|x-y|}{s}\right)\right).$$

But $|\widetilde{Q}_s a(x)|^2 w(x) \frac{ds}{s}\,dx$ is a weighted Carleson measure for all $w \in A_2$ (see [J, pp. 85–87]), so by a standard argument (3.7) is no larger than $\|N(C_{1,100s}f)\|_{2,w}^2$, where N is the nontangential maximal operator $Ng_s(x_0) \equiv$

$\sup_{|x-x_0|<s}|g_s(x)|$. Now C_1 is bounded on L^2 by [M1], and since the kernel $(A(x)-A(y))|x-y|^{-1-\alpha}$ satisfies "standard" Calderon-Zygmund estimates, the corresponding maximal singular integral

$$C_{1,*}f \equiv \sup_{s>0}|C_{1,100s}f|$$

is bounded on L^2_w, $w \in A_2$. Thus, as in [LM, (3.39)], the nontangential maximal function $N(C_{1,100s}f)$ is bounded on L^2_w. In fact, the observation in [LM] holds for any Calderon-Zygmund operator T with "standard" kernel $k(x, y)$, since for $|x-x_0|<s$, we have

$$\begin{aligned}(3.8)\qquad |T_{100s}f(x)| \le & \int |k(x, y)\Phi(|x-y|/s) - k(x_0, y)\Phi(|x_0-y|/s)|\,|f(y)|\,dy \\ &+ T_*f(x_0),\end{aligned}$$

where $\Phi = 1-\varphi$. The first term on the right side of (3.8) is no larger than

$$C\int \frac{s^{\varepsilon}}{(s+|x_0-y|)^{n+\varepsilon}}|f(y)|\,dy \le CMf(x_0),$$

by the standard kernel conditions for $k(x, y)$.

Next, we consider the part of L_s containing the first term in (3.5). We need to estimate

$$(3.9)\qquad \int_{\mathbf{R}}\int_0^\infty \left|\int_{\mathbf{R}} \psi_s(x-z)[A(z)-A(x)]^2\,dz \int_{\mathbf{R}} k_s(x-y)f(y)\,dy\right|^2 \frac{ds}{s^{1+2\alpha}}w(x)\,dx.$$

Now, $|k_s(x-y)| \le C/(|h|+|x-y|)^{1+\alpha}$ for $|h|<s$. Furthermore, $A \in \mathrm{Lip}_\alpha$, so by the change of variables $z \to z+x$, and then $z \to sz$, we have that (3.9) is bounded by

$$(3.10)\qquad \int_{\mathbf{R}}\int_0^\infty \left(\int_{|z|<1} |A(x+sz)-A(x)|P_{s|z|}(|f|)(x)\,dz\right)^2 \frac{ds}{s^{1+2\alpha}}w(x)\,dx,$$

where P_t denotes convolution with the kernel $t^\alpha/(t+|x|)^{1+\alpha}$. Now by Minkowski's integral inequality, the square root of (3.10) is no larger than

$$\int_{|z|<1}\left(\int_{\mathbf{R}}\int_0^\infty |A(x+sz)-A(x)|^2\left(P_{s|z|}(|f|)(x)\right)^2 \frac{ds}{s^{1+2\alpha}}w(x)\,dx\right)^{1/2} dz.$$

The desired estimate now follows by the change of variable $s \to s/|z|$, and a standard argument using the weighted Carleson measure condition (2.4), and the fact that the nontangential maximal function $N(P_sf)$ is bounded on L^2_w, $w \in A_2$.

To conclude the proof of Theorem 1.4, it remains to consider H_s in (3.4). The part of (3.1) corresponding to this term is dominated by

$$\begin{aligned}(3.11)\qquad \int_0^\infty\int_{\mathbf{R}}\Bigg(\int_{\mathbf{R}}\int_{\mathbf{R}} & |\psi_s(x-z)|(A(z)-A(y))^2\frac{s}{|z-y|^{2+\alpha}} \\ & \times \chi\{|z-y|>99s\}|f(y)|\,dy\,dz\Bigg)^2 w(x)\,dx\,\frac{ds}{s^{1+2\alpha}},\end{aligned}$$

where we have applied the mean value theorem to k_s and used the fact that $|x-z|<s$. For $w \in A_2$, again by Muckenhoupt's Theorem, this last expression is no larger than a constant times

$$\int_0^\infty \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{(A(x)-A(y))^2}{|x-y|^{1+2\alpha}} \left(\frac{s}{|x-y|} \right)^{1-\alpha} \right. \\ \left. \times \chi\{|x-y|>99s\}|f(y)|\,dy \right)^2 w(x)\,dx\,\frac{ds}{s}.$$

By Minkowski's integral inequality, we have the bound

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{A(x)-A(y))^2}{|x-y|^{1+2\alpha}} |f(y)| \left(\int_0^{|x-y|/99} \left(\frac{s}{|x-y|} \right)^{2(1-\alpha)} \frac{ds}{s} \right)^{1/2} dy \right)^2 w(x)\,dx \\ \leq C\|C_2(|f|)\|_{2,w}^2,$$

and the theorem follows.

4. Extension to the single layer potential

Consider first the modified single layer potential

$$S_\alpha f(x) \equiv \int_{\mathbb{R}} W_\alpha(x,y) f(y)\,dy,$$

where

$$W_\alpha(x,y) \equiv |x-y|^{\alpha-1} \exp\left[-\frac{(A(x)-A(y)^2}{|x-y|^{2\alpha}} \right].$$

We have the following weighted version of [LM, Theorem 2] (see also [LM, Theorems 4 and 5]).

Theorem 4.1. *Let $A = I_\alpha a$, and suppose $w \in A_2$ if $\frac{1}{2} \leq \alpha < 1$, or $w \in A_1$ if $0 < \alpha < 1$. Then*

$$\|(S_\alpha - c_\alpha I_\alpha) f\|_{I_\alpha(L_w^2)} \leq C_\alpha(\|a\|_*^2 + \|a\|_*^4)\|f\|_{2,w}.$$

Proof of Theorem 4.1. We will follow [LM, Theorem 2] and obtain the theorem by an easy modification of the proof of Theorem 1.4. The operator $S_\alpha - c_\alpha I_\alpha$ has kernel

$$|x-y|^{\alpha-1} \left\{ \exp\left[-\frac{(A(x)-A(y))^2}{|x-y|^{2\alpha}} \right] - 1 \right\}. \tag{4.2}$$

The expression in curly brackets in (4.2) is no larger than a constant times $[A(x)-A(y)]^2|x-y|^{-2\alpha}$, so if we multiply (4.2) by a smooth radial cut-off factor $\varphi(|x-y|/s)$, then we get a term that can be handled exactly like the term corresponding to j_s in (3.2). It therefore remains to treat (4.2) times

$$\Phi\left(\frac{|x-y|}{s} \right) \equiv \left(1 - \varphi\left(\frac{|x-y|}{s} \right) \right).$$

As in (3.4), we must consider the following analogues of (3.47) and (3.48) in [LM]:

$$\int_{\mathbb{R}} \psi_s(x-z) \left\{ \exp\left[-\frac{(A(z)-A(y))^2}{|z-y|^{2\alpha}} \right] - 1 \right\} \\ \times \left[|z-y|^{\alpha-1} \Phi\left(\frac{|z-y|}{s} \right) - |x-y|^{\alpha-1} \Phi\left(\frac{|x-y|}{s} \right) \right] dz, \tag{4.3}$$

$$\int_{\mathbb{R}} \psi_s(x-z)\left\{\exp\left[-\frac{(A(z)-A(y))^2}{|z-y|^{2\alpha}}\right]-\exp\left[-\frac{(A(x)-A(y))^2}{|x-y|^{2\alpha}}\right]\right\} \times |x-y|^{\alpha-1}\Phi\left(\frac{|x-y|}{s}\right)dz. \tag{4.4}$$

These correspond to H_s and L_s in (3.4) respectively. The former can be handled exactly as before; in fact, we obtain the same upper bound (3.11).

Next, by Taylor's theorem the expression in curly brackets in (4.4) equals

$$-\left[\frac{A(z)-A(y))^2}{|z-y|^{2\alpha}}-\frac{(A(x)-A(y))^2}{|x-y|^{2\alpha}}\right]\exp\left[-\frac{(A(x)-A(y))^2}{|x-y|^{2\alpha}}\right], \tag{4.5}$$

$$+\left[\frac{(A(z)-A(y))^2}{|z-y|^{2\alpha}}-\frac{(A(x)-A(y))^2}{|x-y|^{2\alpha}}\right]^2 E(x,y,z), \tag{4.6}$$

where $0 \le E(x,y,z) \le 1$. By analogy to (3.4) and (3.5),

$$\begin{aligned} &\frac{(A(z)-A(y))^2}{|z-y|^{2\alpha}}-\frac{(A(x)-A(y))^2}{|x-y|^{2\alpha}} \\ &= (A(z)-A(y))^2\left[\frac{1}{|z-y|^{2\alpha}}-\frac{1}{|x-y|^{2\alpha}}\right] \\ &\quad+\frac{(A(z)-A(x))^2}{|x-y|^{2\alpha}}+\frac{2(A(z)-A(x))(A(x)-A(y))}{|x-y|^{2\alpha}} \\ &\equiv B_1(x,y,z)+B_2(x,y,z)+B_3(x,y,z). \end{aligned} \tag{4.7}$$

Since $|x-z|<s \ll |x-y|$ (so, in particular, $|x-y| \approx |z-y|$), we can handle the part of (4.5) corresponding to B_1 exactly like H_s in (3.4) (see (3.11)). Since (4.6) is no larger than $CE(x,y,z)\sum_{i=1}^{3}(B_i(x,y,z))^2$, and since trivially

$$|B_1| \le C\|A\|^2_{\mathrm{Lip}\,\alpha} \le C\|a\|^2_*,$$

the same reasoning applies to the parts of (4.6) corresponding to B_1. Similarly, those parts of (4.5) and (4.6) involving B_2 may be treated exactly like the first term in (3.5) (see (3.9), (3.10), and the related discussion). The latter argument also applies to the term $(B_3(x,y,z))^2E(x,y,z)$ arising in (4.6).

Thus, it remains only to consider the following part of (4.5): $B_3(x,y,z)$ $\times\exp[-(A(x)-A(y))^2/|x-y|^{2\alpha}]$ (we have ignored multiplication by -2). This is the only term where we do not reduce matters to the treatment of an appropriate positive operator, so the presence of a bounded, nonconstant multiplicative factor can no longer be ignored. If we plug this last expression into (4.4) in place of $\{\ \}$ and let the corresponding operator act on a function f, then we get (since $\int\psi=0$)

$$Q_sA(x)T_{100s}f(x),$$

where T_{100s} has the (truncated) standard kernel

$$\frac{(A(x)-A(y))}{|x-y|^{1+\alpha}}\exp\left[-\frac{(A(x)-A(y))^2}{|x-y|^{2\alpha}}\right]\Phi\left(\frac{|x-y|}{s}\right).$$

As before (see (3.7), (3.8), and the related discussion), the theorem will follow by weighted Carleson measure theory once we show that the maximal singular integral

$$T_*f \equiv \sup_{s>0}|T_{100s}f|$$

is bounded on L^2_w, $w \in A_2$. But this is easy since the mean value theorem gives

$$\exp\left[-\frac{(A(x)-A(y))^2}{|x-y|^{2\alpha}}\right] = 1 + \left[\frac{(A(x)-A(y))^2}{|x-y|^{2\alpha}}\right]\widetilde{E}(x, y), \tag{4.8}$$

with $|\widetilde{E}| \leq 1$. The term corresponding to 1 is just the first fractional commutator C_1, and the term corresponding to the second part of the right side of (4.8) is no larger than $C\|A\|_{\mathrm{Lip}\,\alpha} C_2(|f|)$, and we are done.

In conclusion, we remark that as in [LM], a straightforward modification of the above arguments enables one to multiply the kernels that we have considered (e.g., (4.2) or (1.3)) by $\chi\{x-y>0\}$. In particular, for $\alpha = \frac{1}{2}$, we can treat the boundary single layer potential for the heat equation for all $w \in A_2$ and thus for all p, $1 < p < \infty$.

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