

THE NULL SET OF THE FOURIER TRANSFORM FOR A SURFACE CARRIED MEASURE

LIMIN SUN

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ABSTRACT. Let du be a smooth positive measure carried by a smooth compact hypersurface S that is strictly convex and without boundary in R^n ($n \geq 2$). Assume that both S and du are symmetric about the origin. If \hat{du} denotes the Fourier transform of du then we show that the null set of \hat{du} is a disjoint union of a compact set and countably many hypersurfaces that are all diffeomorphic to the unit sphere S^{n-1} .

1. INTRODUCTION

The Fourier transforms of surface carried measures play an important role in many research areas related to harmonic analysis such as the restriction theorems and the maximal averages over hypersurfaces (cf. [2] and the literature cited therein). It turns out that the geometric property of the hypersurface is crucial to those investigations.

For a smooth measure du on some smooth hypersurfaces s in R^n , the Fourier transform of du is defined by

$$\hat{du}(y) = \int_S e^{-ix \cdot y} du(x), \quad y \in R^n.$$

Let $N(S, du) = \{y \in R^n; \hat{du}(y) = 0\}$ be the null set of \hat{du} . If $d\sigma$ is the standard surface measure on the unit sphere S^{n-1} , then

$$\hat{d\sigma}(y) = (2\pi)^{n/2} J_{(n-2)/2}(|y|) \cdot |y|^{-(n-2)/2}.$$

Hence, the null set of $\hat{d\sigma}$ consists of countably many hyperspheres in R^n , i.e.,

$$N(S^{n-1}, d\sigma) = \bigcup_{j=1}^{\infty} \{y \in R^n; |y| = r_j\} \quad (\text{disjoint union}),$$

where r_j ($j = 1, 2, \dots$) is the enumeration of the positive zeros of the Bessel function $J_{(n-2)/2}(r)$.

From the geometric point of view, it is natural to ask what happens to $N(S, du)$ if du is a general smooth positive measure on some smooth hypersurface S with nonzero Gaussian curvature.

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In this paper, the smooth hypersurface S is assumed to be strictly convex (i.e., with positive Gaussian curvature) and without boundary. Assume further that both S and du are symmetric about the origin (i.e., $x \in S$ implies $-x \in S$, and $du(E) = du(-E)$ for any measurable set $E \subset S$). The main result is as follows.

Theorem. *Let du be a smooth positive measure carried by a smooth compact hypersurface S in R^n ($n \geq 2$). Suppose that S and du satisfy the assumptions stated above. Then $N(S, du) = N_0 \cup (\bigcup_{j=1}^{\infty} N_j)$ (disjoint union), where N_0 is a compact set and N_j ($j = 1, 2, \dots$) is diffeomorphic to the unit sphere S^{n-1} .*

2. PRELIMINARIES

In order to prove the theorem, we should understand the principal part of $\hat{d}u$ as well as that of the derivative of $\hat{d}u$ in radial direction. For this purpose, we need an exact formula of $\hat{d}u$. For $y \in R^n$, write $r = |y|$ and $\theta = y/|y|$. The following result is due to Greenleaf [1].

Lemma 1. *Let S and du be given as in the theorem. Then*

$$\hat{d}u(y) = C_n \cdot d(\theta)^{n-2} \cdot [(t^{-(n-2)/2} J_{(n-2)/2}(t)) * \hat{g}(\theta, t)](rd(\theta)),$$

where C_n is a constant, $d(\theta) \in C^\infty(S^{n-1})$, $g(\theta, s) \in C^\infty(S^{n-1} \times R)$, and $g(\theta, s) \in C_0^\infty(R)$ for each fixed $\theta \in S^{n-1}$; $\hat{g}(\theta, t)$ is the Fourier transform of $g(\theta, s)$ about the variable s ; $*$ denotes the convolution about the variable t .

Remark 1. Lemma 1 is a particular case of Lemma 6 in [1], which deals with general smooth measures (without the restriction of positiveness and symmetry).

Proposition 1. *Let $d(\theta)$ and $g(\theta, s)$ be as in Lemma 1. Then $d(\theta) > 0$ for all $\theta \in S^{n-1}$ and $g(\theta, s) \neq 0$ for all $(\theta, s) \in S^{n-1} \times [-1, 1]$.*

Proof. From the proof of Lemma 6 in [1], it is obvious that $d(\theta) > 0$. Moreover, $g(\theta, s)$ is an extension of the function $\tilde{f}(\theta; d(\theta)s)$. In fact, for $(\theta, s) \in S^{n-1} \times [-1, 1]$,

$$(1) \quad g(\theta, s) = \tilde{f}(\theta; d(\theta)s) = \int_{S^{n-2}} f(\theta; d(\theta)s, x')^{-1} \frac{du}{dv} d\theta(x'),$$

where dv is the induced Lebesgue measure on S , $d\sigma(x')$ is the standard surface measure on S^{n-2} , and f is smooth and nonzero (cf. [1, pp. 529–530]). Since we assume that du is positive, the three measures that appeared in (1) are all positive. Thus, the conclusion follows.

Proposition 2. *The value of $g(\theta, s)$ can be modified such that $g(\theta, s)$ is an even function of s and $g(\theta, s) = 0$ for all $\theta \in S^{n-1}$ and $|s| \geq 2$. Meanwhile, the conclusion of Lemma 1 remains true.*

Proof. Again from the proof of Lemma 6 in [1], one easily sees that the central symmetricity of S and du ensures that $\tilde{f}(\theta; d(\theta)s)$ is an even function of s . Moreover, for each fixed $\theta \in S^{n-1}$, $g(\theta, s)$ is an arbitrary extension of $\tilde{f}(\theta; d(\theta)s)$ ($s \in [-1, 1]$) to an element of $C_0^\infty(R)$. Hence, we can easily make $g(\theta, s)$ be even about s . Choose an even function $k(s) \in C_0^\infty(R)$ such

that $k(s) = 1$ for $s \in [-1, 1]$ and $k(s) = 0$ for $|s| \geq 2$. Then the conclusion of Lemma 1 remains true when $g(\theta, s)$ is replaced by $g(\theta, s)k(s)$. This completes the proof of Proposition 2.

Let $\mathcal{S}(R)$ be the Schwartz class of rapidly decreasing functions on R .

Lemma 2. Suppose that $h(t) \in \mathcal{S}(R)$ is an even function. For $p \geq q \geq 0$, let $H(r) = [(t^{-q} J_p(t)) * h(t)](r)$, where J_p is the p th Bessel function. Then there exists a constant C_1 independent of r such that

$$H(r) = \left(\frac{2}{\pi}\right)^{1/2} \cdot r^{-(q+1/2)} \left[\hat{h}(1) \cos\left(r - \frac{1}{2}\pi p - \frac{1}{4}\pi\right) + R_1(r) \right]$$

and $|R_1(r)| \leq C_1 r^{-1}$ for $r > 1$, where \hat{h} is the Fourier transform of h .

Proof. Since $h(t) \in \mathcal{S}(R)$, $C_{q,h} = \sup_{t \in R} |h(t)| \cdot |t|^{(q+3)} < \infty$. Then

$$(2) \quad \int_{|t|>r/2} |h(t)| dt \leq C_{q,h} \cdot \int_{|t|>r/2} |t|^{-(q+3)} dt \leq C_q \cdot C_{q,h} \cdot r^{-(q+2)} \quad \text{for } r > 0$$

and

$$(3) \quad \begin{aligned} \left(\int_{|t|>r/2} |h(t)|^{4/3} dt \right)^{3/4} &\leq C_{q,h} \left(\int_{|t|>r/2} |t|^{-4(q+3)/3} dt \right)^{3/4} \\ &\leq C'_q \cdot C_{q,h} \cdot r^{-(q+9/4)} \\ &\leq C'_q \cdot C_{q,h} \cdot r^{-(q+3/2)} \quad \text{for } r > 1. \end{aligned}$$

Write $a_p = \frac{1}{2}\pi p + \frac{1}{4}\pi$. Since $h(t)$ is even, $\hat{h}(1) = \int_{-\infty}^{\infty} h(t) \cos t dt$. Thus we have

$$(4) \quad \begin{cases} H(r) - \left(\frac{2}{\pi}\right)^{1/2} \cdot \hat{h}(1) r^{-(q+1/2)} \cos(r - a_p) = I_1 + I_2 + I_3, \\ I_1 = \int_{|t| \leq r/2} [(r-t)^{-q} \cdot J_p(r-t) \\ \quad - \left(\frac{2}{\pi}\right)^{1/2} r^{-(q+1/2)} \cos(r - a_p) \cos t] h(t) dt, \\ I_2 = \int_{|t| > r/2} (r-t)^{-q} J_p(t) h(t) dt, \\ I_3 = - \left(\frac{2}{\pi}\right)^{1/2} r^{-(q+1/2)} \cos(r - a_p) \cdot \int_{|t| > r/2} h(t) \cos t dt. \end{cases}$$

It follows from (2) that for $r > 1$,

$$(5) \quad \begin{aligned} |I_3| &\leq \left(\frac{2}{\pi}\right)^{1/2} r^{-(q+1/2)} \int_{|t|>r/2} |h(t)| dt \\ &\leq C_q \cdot C_{q,h} r^{-(q+1/2)} r^{-(q+2)} \leq C_q \cdot C_{q,h} r^{-(q+3/2)}. \end{aligned}$$

An application of Hölder's inequality together with (3) yields

$$(6) \quad \begin{aligned} |I_2| &\leq \left(\int_{|t|>r/2} |(r-t)^{-q} \cdot J_p(r-t)|^4 dt \right)^{1/4} \left(\int_{|t|>r/2} |h(t)|^{4/3} dt \right)^{3/4} \\ &\leq C'_p \cdot C_{p,h} \cdot \left(\int_{-\infty}^{\infty} |t^{-q} \cdot J_p(t)|^4 dt \right)^{1/4} r^{-(q+3/2)}, \quad r > 1. \end{aligned}$$

For $p \geq q \geq 0$, the well-known fact that

$$\lim_{t \rightarrow 0} J_p(t)/t^p < \infty \quad \text{and} \quad \overline{\lim}_{t \rightarrow \infty} |J_p(t)t^{1/2}| < \infty$$

implies $\int_0^\infty |t^{-q} J_p(t)|^4 dt < \infty$. Summarizing the above discussion, we get

$$(7) \quad |I_2| + |I_3| \leq C(p, q, h) \cdot r^{-(q+3/2)} \quad \text{for } r > 1.$$

Next recall the asymptotic formula $J_p(t) = (2/\pi)^{1/2} t^{-1/2} \cos(t - a_p) + Q(t)$, where $|Q(t)| \leq C_p \cdot |t|^{-3/2}$ for all $t \in \mathbb{R}$ and C_p is a constant dependent on p alone. Then, we have

$$(8) \quad \begin{cases} I_1 = I'_1 + I'_2 + I'_3, \\ I'_1 = \left(\frac{2}{\pi}\right)^{1/2} \int_{|t| \leq r/2} [(r-t)^{-(q+1/2)} - r^{-(q+1/2)}] \cos(r-t-a_p) h(t) dt, \\ I'_2 = \left(\frac{2}{\pi}\right)^{1/2} r^{-(q+1/2)} \cdot \int_{|t| \leq r/2} [\cos(r-t-a_p) - \cos(r-a_p) \cos t] h(t) dt, \\ I'_3 = \int_{|t| \leq r/2} (r-t)^{-q} Q(r-t) h(t) dt. \end{cases}$$

But $|r-t| \geq r/2$ for $|t| \leq r/2$. Hence,

$$(9) \quad |I'_3| \leq C_p \left(\frac{1}{2}r\right)^{-(q+3/2)} \int_{-\infty}^\infty |h(t)| dt.$$

Note that h is even. A direct calculation shows

$$(10) \quad I'_2 = -\left(\frac{2}{\pi}\right)^{1/2} \sin(r-a_p) r^{-(q+1/2)} \int_{|t| \leq r/2} h(t) \sin t dt = 0.$$

Furthermore, it is easy to check that for $|t| \leq \frac{1}{2}r$,

$$\left| \left(1 - \frac{t}{r}\right)^{-(q+1/2)} - 1 \right| \leq \left(q + \frac{1}{2}\right) q^{(q+1/2)} \cdot \frac{|t|}{r}.$$

This fact yields

$$(11) \quad \begin{aligned} |I'_1| &\leq \left(\frac{2}{\pi}\right)^{1/2} \cdot r^{-(q+1/2)} \int_{|t| \leq r/2} \left| \left(1 - \frac{t}{r}\right)^{-(q+1/2)} - 1 \right| \cdot |h(t)| dt \\ &\leq \left(q + \frac{1}{2}\right) 2^{(q+1/2)} \cdot r^{-(q+3/2)} \int_{-\infty}^\infty |th(t)| dt. \end{aligned}$$

From (8)–(11), we get

$$(12) \quad |I_1| \leq \tilde{C}(p, q, h) r^{-(q+3/2)} \quad \text{for } r > 0.$$

Lemma 2 follows from (4), (7), and (12) immediately.

Lemma 3. Let $h(t)$ be given as in Lemma 2. For $p \geq 0$, let $H(r) = [(t^{-p} \cdot J_p(t)) * h(t)](r)$. Then

$$\frac{dH(r)}{dr} = -\left(\frac{2}{\pi}\right)^{1/2} \cdot r^{-(p+1/2)} \cdot \left[\hat{h}(1) \sin\left(r - \frac{1}{2}\pi p - \frac{1}{4}\pi\right) + R_2(r) \right]$$

and $|R_2(r)| \leq C_2 \cdot r^{-1}$ for $r > 1$, where C_2 is a constant independent of r .

Proof. Note that

$$\frac{dH(r)}{dr} = \left[\left(\frac{d}{dt}(t^{-p} \cdot J_p(t)) \right) * h(t) \right](r) \quad \text{and} \quad \frac{d}{dt}(t^{-p} \cdot J_p(t)) = -t^{-p} \cdot J_{p+1}(t).$$

Since $\cos(r - \frac{1}{2}\pi(p+1) - \frac{1}{4}\pi) = \sin(r - \frac{1}{2}\pi p - \frac{1}{4}\pi)$, Lemma 3 is a direct consequence of Lemma 2.

Remark 2. The basic knowledge about the Bessel function used in the proofs of Lemmas 2 and 3 can be found in Watson's book [3].

3. PROOF OF THE THEOREM

Let $H(\theta, r) = [(t^{-(n-2)/2} J_{(n-2)/2}(t)) * \hat{g}(\theta, t)](r)$, where g is given as in Lemma 1. For each fixed θ , since $g(\theta, s) \in C_0^\infty(R)$ is even (see Proposition 2), $\hat{g}(\theta, t) \in \mathcal{S}(R)$ is also even. Applying Lemmas 2 and 3 we get

$$\begin{aligned} H(\theta, r) &= \left(\frac{2}{\pi} \right)^{1/2} r^{-(n-1)/2} \\ (13) \quad &\cdot \left[\hat{g}(\theta, 1) \cos \left(r - \frac{1}{4}\pi(n-2) - \frac{1}{4}\pi \right) + R_1(\theta, r) \right] \\ &= \left(\frac{2}{\pi} \right)^{1/2} r^{-(n-1)/2} \left[g(\theta, -1) \cos \left(r - \frac{1}{4}\pi n + \frac{1}{4}\pi \right) + R_1(\theta, r) \right] \end{aligned}$$

and

$$\begin{aligned} (14) \quad \frac{\partial H(\theta, r)}{\partial r} &= - \left(\frac{2}{\pi} \right)^{1/2} r^{-(n-1)/2} \\ &\cdot \left[g(\theta, -1) \sin \left(r - \frac{1}{4}\pi n + \frac{1}{4}\pi \right) + R_2(\theta, r) \right]. \end{aligned}$$

Moreover, there exist constants $C_1(\theta)$ and $C_2(\theta)$ independent of r such that $|R_1(\theta, r)| \leq C_1(\theta)r^{-1}$ and $|R_2(\theta, r)| \leq C_2(\theta)r^{-1}$ for $r > 1$.

Proposition 3. *There exists a constant C_0 independent of θ such that $|C_1(\theta)| \leq C_0$ and $|C_2(\theta)| \leq C_0$ for all $\theta \in S^{n-1}$.*

Proof. We easily see from the proof of Lemma 2 that, besides the dimension n , $C_1(\theta)$ and $C_2(\theta)$ depend only on

$$(15) \quad \sup_{t \in R} |\hat{g}(\theta, t)| \cdot |t|^{(n/2+2)}, \quad \int_{-\infty}^{\infty} |\hat{g}(\theta, t)| dt, \quad \int_{-\infty}^{\infty} |t \hat{g}(\theta, t)| dt.$$

Recall that, for any fixed θ , $\hat{g}(\theta, t)$ is the Fourier transform of $g(\theta, s)$ whose support lies in an interval independent of θ (see Proposition 2). This fact implies that the first term in (15) is uniformly bounded about θ , and hence the two integrals in (15) are also uniformly bounded. The proof of Proposition 3 is complete.

On the other hand, $g(\theta, -1) \neq 0$ (see Proposition 1) and $g(\theta, -1) \in C^\infty(S^{n-1})$ implies $\text{Min}(|g(\theta, -1)|) \neq 0$. This fact, together with Proposition 3 combining (13) and (14), shows that there exists an absolute constant $R_0 > 0$ such that for any fixed $\theta \in S^{n-1}$,

$$\{r > R_0; H(\theta, r) = 0\} = \{r_j(\theta); j = 1, 2, \dots\},$$

where $r_{j_1}(\theta) \neq r_{j_2}(\theta)$ ($j_1 \neq j_2$) and

$$\left. \frac{\partial H(\theta, r)}{\partial r} \right|_{r=r_j(\theta)} \neq 0, \quad j = 1, 2, \dots$$

Apparently, $H(\theta, r) \in C^\infty(S^{n-1} \times R)$. Then the implicit function theorem asserts that $r_j(\theta) \in C^\infty(S^{n-1})$.

Let $R'_0 = R_0 / \text{Min}(d(\theta))$. According to Lemma 1 and the above discussion, we know that

$$N(S, du) \cap \{y \in R^n; |y| > R'_0\} \subseteq \bigcup_{j=1}^{\infty} \left\{ \frac{r_j(\theta)}{d(\theta)} \theta; \theta \in S^{n-1} \right\}.$$

Obviously, $r_j(\theta)\theta/d(\theta)$ ($\theta \in S^{n-1}$) is a hypersurface diffeomorphic to S^{n-1} . We conclude the proof of the theorem by mentioning that $N(S, du) \cap \{y \in R^n; |y| \leq R'_0\}$ is a compact set.

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