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FLAT CORE PROPERTIES ASSOCIATED TO THE p-LAPLACE OPERATOR

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ABSTRACT. We study the formation of a flat hat pattern in the profile of the positive solution of an equation of the type: $\varepsilon\Delta_p u - u^{p-1}(1-u)^\theta = 0$ $(0 < \theta < p-1)$ in a bounded domain Ω. When ε tends to 0^+, the growth of the zone where $u = u_\varepsilon$ takes the value 1 in Ω is studied.

INTRODUCTION AND STATEMENT OF THE RESULTS

This paper deals with the study of the limit behaviour when ε tends to 0^+ of the shape of the positive solution $u = u_\varepsilon$ of the problem

$$\begin{aligned} -\varepsilon\Delta_p u + f(u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a connected, bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with a C^2 boundary $\partial\Omega$; Δ_p is the p-Laplace operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \tag{2}$$

with $p > 1$; and f is continuous with nonpositive values. Such a problem appears when studying the stationary states of a strongly nonlinear heat equation in an absorbing-reacting media (see [D] for physical examples and further references). The precise hypotheses on f are the following:

(H1) f is continuous on $[0, \infty)$ and $r \mapsto f(r)/r^{p-1}$ is increasing.
(H2) $\lim_{r\downarrow 0} f(r)/r^{p-1} = -1$.
(H3) There exist $C > 0$ and $\theta \in (0, p-1)$ such that $\lim_{r\uparrow 1} f(r)/(1-r)^\theta = -C$.

The specific phenomenon we shall study is the formation of a flat hat pattern inside Ω, that is, a zone where u takes the value 1 and the growth of this zone when ε tends to 0.

The typical example of a function f satisfying (H1)–(H3) is $f(u) = u^{p-1} - u^q$, thus problem (1) becomes

$$\begin{aligned} -\varepsilon\Delta_p u &= u^{p-1} - u^q \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

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If we set $\varepsilon = 1/\lambda$ and $u = v\varepsilon^{1/(q+1-p)}$, then (3) reads as

$$(4) \qquad \begin{aligned} -\Delta_p v &= \lambda v^{p-1} - v^q \quad && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The appearance of the flat zone for the solution of (4) for large λ was first observed by Guedda and Veron. These authors in [GV] studied the structure of the set of solutions of the nonlinear eigenvalue problem

$$(5) \qquad \begin{aligned} -(|v_x|^{p-2}v_x)_x &= \lambda|v|^{p-2}v - |v|^{q-1}v \quad \text{in } (0, 1), \\ v(0) &= v(1) = 0. \end{aligned}$$

It is proved in [GV] that for

$$(6) \qquad q > p - 1 > 1$$

and λ large enough, the unique positive solution of (5) satisfies $v(x) = \lambda^{1/(q+1-p)}$ for $x \in [x(\lambda), 1 - x(\lambda)]$ where $x(\lambda) > 0$ and $x(\lambda) \sim C\lambda^{-1/p}$ at infinity. Another consequence described in [GV] is that for λ large enough, the set of solutions v of (5) with $k-1$ simple zeros on $(0, 1)$ and $v_x(0) > 0$ is homeomorphic to the $(k-1)$-dimensional unit cube. P. L. Lions asked one of the authors whether such phenomenon still existed for the N-dimensional case.

If we define

$$(7) \qquad \lambda_1 = \min\left\{\int_\Omega |\nabla u|^p\,dx \Big/ \int_\Omega |u|^p\,dx : u \in W_0^{1,p}(\Omega)\backslash\{0\}\right\},$$

it is a classical fact that under condition (6), for any $\lambda > \lambda_1$ there exists v positive in Ω satisfying (4). As for problem (1) we know from [DS] that if $\varepsilon < 1/\lambda_1$ and f satisfies (H1), (H2), then there exists a unique $u - u_\varepsilon$ belonging to $C^1(\overline{\Omega})$ which is a positive in Ω solution of (1). Moreover if (H3) holds then u takes its values in $[0, 1]$. If we define

$$(8) \qquad \Omega_\lambda = \{x \in \Omega : v(x) \equiv \lambda^{1/(q+1-p)}\}$$

then Ω_λ is a compact, possibly empty, subset of Ω. We have the following answer to Lions's question

Theorem 1. *Assume* (6), $\lambda > \lambda_1$, v *is the positive solution of* (4), *and* Ω_λ *is defined by* (8). *Then there exists* $\lambda^* = \lambda^*(\Omega, p, q) > \lambda_1$ *such that:*

(i) *if* $\lambda < \lambda^*$ *the set* Ω_λ *is empty;*

(ii) *if* $\lambda \geq \lambda^*$ *the set* Ω_λ *is not empty and*

$$(9) \qquad \operatorname{dist}(\Omega_\lambda, \partial\Omega) \leq C\lambda^{-1/p},$$

where $C = C(\Omega, p, q) > 0$.

Theorem 1 is a consequence of

Theorem 2. *Assume* (H1)–(H3) *with* $p > 1$. *Then for* $\varepsilon > 0$ *small enough the coincidence set* Ω_ε *of the solution* u *of* (1) *defined by*

$$(10) \qquad \Omega_\varepsilon = \{x \in \Omega : u(x) = 1\}$$

is not empty and there exists a constant $C > 0$ *such that*

$$(11) \qquad \operatorname{dist}(\Omega_\varepsilon, \partial\Omega) \leq C\varepsilon^{1/p}.$$

PROOFS OF THE RESULTS

We first extend the function f on $(-\infty, 0)$ such that the resulting function defined on R is a continuous odd function. This function is still denoted by f.

Lemma 1. *Let w_1 and w_2 be two functions belonging to $C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ and such that*

$$0 = w_1 \le w_2 \quad \text{on } \partial\Omega \tag{12}$$

and

$$w_1 \le w_2, \tag{13}$$
$$-\Delta_p w_1 + f(w_1) \le 0, \tag{14}$$
$$-\Delta_p w_2 + f(w_2) \ge 0 \tag{15}$$

in Ω. Then there exists a function w in $C_0(\Omega) \cap W^{1,p}(\Omega)$ satisfying

$$w_1 \le w \le w_2, \tag{16}$$
$$-\Delta_p w + f(w) = 0 \tag{17}$$

in Ω.

This result is due to Deuel and Hess [DeH] and extends previous results of Amann, Sattinger, and others (see [A] for example).

Lemma 2. *Let $w \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ be a positive solution of (17) in Ω. Then for $C > 1$ (resp. $0 < C < 1$) we have*

$$-\Delta_p(Cw) + f(Cw) \ge 0 \quad (\textit{resp. } -\Delta_p(Cw) + f(Cw) \le 0) \tag{18}$$

in Ω.

Proof. For $C > 1$ we have

$$\Delta_p(Cw) = C^{p-1}\Delta_p w = C^{p-1} f(w) = (Cw)^{p-1} f(w)/w^{p-1}. \tag{19}$$

From (H1) we have $f(w)/w^{p-1} \le f(Cw)/(Cw)^{p-1}$, which yields (18). The same proof applies for $0 < C < 1$.

Lemma 3. *Assume* (Hi) (i = 1, 2, 3) *and let $u = u_\varepsilon$ be the positive solution of* (9). *Then u_ε converges to 1 as ε tends to 0, uniformly on any compact subset K of Ω.*

Proof. By the maximum principle, $u_\varepsilon \le 1$ in $\overline{\Omega}$. The intent of this proof is to construct a subsolution v of (9) with the form

$$v = 1 - e^{-\psi/\varepsilon'}, \qquad \varepsilon' = \varepsilon^{1/p}, \tag{20}$$

with some $\psi > 0$ in Ω, vanishing on $\partial\Omega$; the function ψ will be made precise later. Then

$$\nabla v = \frac{1}{\varepsilon'} e^{-\psi/\varepsilon'} \nabla\psi$$

and

$$-\Delta_p v = (\varepsilon')^{-p} e^{-(p-1)\psi/\varepsilon'} \{(p-1)|\nabla\psi|^p - \varepsilon' \Delta_p \psi\},$$

which yields

$$-\varepsilon\Delta_p v + f(v) = [(p-1)|\nabla\psi|^p - \varepsilon'\Delta_p\psi] e^{-(p-1)\psi/\varepsilon'} + f(v). \tag{21}$$

Set $E_1 = (p-1)|\nabla\psi|^p$, $E_2 = -\varepsilon'\Delta_p\psi$, and $y = e^{-\psi/\varepsilon'}$. We claim that for ε' small enough

$$E_1 + E_2 \leq -y^{1-p} f(1-y). \tag{22}$$

For $\delta > 0$ we define

$$\Omega_-^\delta = \{x\in\Omega : y\leq\delta\} = \{x\in\Omega : \psi\geq\varepsilon'\ln(1/\delta)\},$$
$$\Omega_+^\delta = \{x\in\Omega : y>\delta\} = \{x\in\Omega : \psi<\varepsilon'\ln(1/\delta)\}.$$

From (H3) $\lim_{y\downarrow 0^+}(-y^{-\theta}f(1-y)) = C$; henceforth there exists $\delta_0\in(0,1)$ such that $-y^{-\theta}f(1-y) > C/2$ for $0<y<\delta_0$, which implies

$$-\frac{1}{y^{p-1}}f(1-y) > \frac{C}{2\delta^{p-1-\theta}} \quad \forall y\in(0,\delta_0) \tag{23}$$

as $p-1-\theta>0$. We shall take $\psi=\phi_1^p$ where ϕ_1 is the unique positive solution with upper bound 1 of

$$\begin{aligned} -\Delta_p\phi_1 &= \lambda_1\phi_1^{p-1} \quad &&\text{in } \Omega,\\ \phi_1 &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{24}$$

There exists $M>0$ such that for any $\varepsilon'\in(0,1]$ we have

$$|(p-1)|\nabla\psi|^p - \varepsilon'\Delta_p\psi| \leq M, \tag{25}$$

and there exists $\delta_1\in(0,\delta_0]$ such that for $\delta<\delta_1$

$$(p-1)|\nabla\psi|^p - \varepsilon'\Delta_p\psi \leq M \leq c/2\delta^{p-1-\theta} \tag{26}$$

in Ω, which implies that (22) holds in Ω_-^δ.

For the estimate in Ω_+^δ note that there exist two positive constants $l(\delta)$ and $r(\delta)$ such that

$$-f(1-y) \geq l(\delta)(1-y)^{p-1} \quad \forall y\in(\delta,1) \tag{27}$$

and, consequently,

$$-f(1-y) \geq r(\delta)(\psi/\varepsilon')^{p-1} \tag{28}$$

if $\delta<y\leq 1$ or $\psi/\varepsilon' < \ln(1/\delta)$; we used here that $(1-e^{-\rho})/\rho$ is bounded below on $(0, \ln(1/\delta))$. In order to have

$$E_1 \leq -f(1-y)/y^{p-1} \tag{29}$$

in Ω_+^δ, it is sufficient to assure (with $y\leq 1$) that

$$(p-1)|\nabla\psi|^p \leq r(\delta)(\psi/\varepsilon')^{p-1} \tag{30}$$

or, equivalently,

$$(\varepsilon')^{p-1}|\nabla\psi|^p \leq \frac{r(\delta)}{p-1}\psi^{p-1}. \tag{31}$$

As $\psi=\phi_1^p$ we have

$$(\varepsilon')^{p-1}|\nabla\psi|^p\psi^{1-p} = (\varepsilon')^{p-1}p^p|\nabla\phi_1|^p. \tag{32}$$

For $\delta\in(0,\delta_1)$ fixed, we can choose $\varepsilon'_0>0$ such that (31) holds for $0<\varepsilon'\leq\varepsilon'_0$.

For the remaining term we have

$$\Delta_p \psi = \Delta_p \phi_1^p = p^{p-1}(p-1)^2 \phi_1^{(p-1)^2-1} |\nabla \phi_1|^P - \lambda_1 p^{p-1} \phi_1^{p(p-1)} .$$

As $\partial \phi_1 / \partial \nu < 0$ on $\partial \Omega$ there exists a neighborhood $\mathbf{D}$ of $\partial\Omega$ such that

$$(p-1)^2 |\nabla \phi_1|^p > \lambda_1 \phi_1^p \tag{33}$$

in $\mathbf{D}$. For δ fixed in $(0, \delta_1)$ there exists $\varepsilon_1' \in (0, \varepsilon_0')$ such that for any $\varepsilon' \in (0, \varepsilon_1')$, $\Omega_+^\delta \subset \mathbf{D}$. For such a limitation on ε' we have $\Delta_p \psi > 0$ in Ω_+^δ, which implies

$$E_1 + E_2 \le -y^{1-p} f(1-y) \tag{34}$$

in Ω_+^δ. Henceforth, with this restriction on ε', v satisfies

$$-\varepsilon' \Delta_p v + f(v) \le 0 \tag{35}$$

in Ω and v vanishes on $\partial\Omega$. Now we compare u and v. By Vazquez's maximum principle [V], $\partial u / \partial \nu < 0$ on $\partial \Omega$; therefore, there exists $C > 1$ such that $Cu \ge v$ in Ω. Using Lemmas 2 and 1 we get that there exists a solution u^* of (9) such that $v \le u^* \le Cu$. By uniqueness $u^* = u \ge v$. For any compact subset $K \subset \Omega$, there exists $\eta(K) > 0$ such that $\psi \ge \eta(K)$ on K. Letting ε tend to 0 implies the claimed result.

Proof of Theorem 2. Let $\tilde{u} = \tilde{u}_\varepsilon = 1 - u$. From (H3) there exists $\delta_0 > 0$ such that $-\tilde{u}^{-\theta} f(1-\tilde{u}) > c/2$ for $0 < \tilde{u} \le \delta_0$, which yields

$$-\Delta_p \tilde{u} + \frac{c}{2} \tilde{u}^\theta \le 0 \tag{36}$$

if $0 < \tilde{u} \le \delta_0$. For $\eta > 0$ let K_η be the subset of the x's in Ω such that $\operatorname{dist}(x, \partial\Omega) \ge \eta$. From Lemma 3, for any $\delta \in (0, \delta_0)$ there exists $\varepsilon(\delta) > 0$ such that for $0 < \varepsilon < \varepsilon(\delta)$ we have

$$\max\{\tilde{u}_\varepsilon(x) : x \in K_\eta\} < \delta . \tag{37}$$

Let $h = h_\delta$ be the solution of

$$\begin{aligned} -\Delta_p h + \frac{c}{2} h^\theta &= 0 \quad \text{in } B_\eta(0), \\ h &= \delta \quad \text{on } \partial B_\eta(0). \end{aligned} \tag{38}$$

We know from Diaz-Herrero's paper [DH] (see also [D, p. 41]) that there exists $\delta > 0$ such that $h_\delta(0) = 0$. Let $x_0 \in K_{2\eta}$. By comparison, $\tilde{u}(x) \le h_\delta(x - x_0)$ for $|x - x_0| < \eta$. Thus $\tilde{u}(x_0) = 0$. Therefore $\tilde{u}(x) \equiv 0$ on $K_{2\eta}$.

In order to obtain the final estimate we use a local scaling argument. As $\partial\Omega$ is C^2 there exists $\rho > 0$ such that for any $a \in \partial\Omega$ the open ball with center $a - \rho \overrightarrow{\nu}_a$ and radius ρ is included into Ω ($\overrightarrow{\nu}_a$ is the normal unit vector to $\partial\Omega$ at a). As we already proved, there exists $\varepsilon_1 > 0$ such that the positive solution z of

$$\begin{aligned} -\varepsilon_1 \Delta_p z + f(z) &= 0 \quad \text{in } B_\rho(0), \\ z &= 0 \quad \text{on } \partial B_\rho(0), \end{aligned} \tag{39}$$

is such that $z(x) \equiv 1 \quad \forall x \in B_{\rho/2}(0)$. For $k > 0$ the function z_k defined by $z_k(x) = z(kx)$ satisfies

$$\begin{aligned} -\varepsilon_1 k^{-p} \Delta_p z_k + f(z_k) &= 0 \quad \text{in } B_{\rho/k}(0), \\ z_k &= 0 \quad \text{on } \partial B_{\rho/k}(0) \end{aligned} \tag{40}$$

and is such that $z_k(x) \equiv 1 \ \forall x \in B_{\rho/2k}(0)$. For $0 < \varepsilon < \varepsilon_1$ let k be $(\varepsilon_1/\varepsilon)^{1/p}$, $k > 1$. For any $a \in \Omega$ such that $\operatorname{dist}(a, \partial\Omega) \geq \rho/k$ we can compare $u(x)$ and $z_k(x-a)$ in $B_{\rho/k}(a)$. By the same way as in the proof of Lemma 3, we use Lemmas 2 and 1 with $\alpha > 0$ small enough. We get

$$\alpha z_k(x-a) \leq u(x) \quad \text{in } B_{\rho/k}(a), \tag{41}$$

which implies

$$z_k(x-a) \leq u(x) \quad \text{in } B_{\rho/k}(a). \tag{42}$$

We deduce that $u \equiv 1$ in $B_{\rho/2k}(a)$, which implies (11).

Remark 1. It is clear that the coincidence set Ω_ε may be empty if ε is too large. To have an estimate of this minimal ε we can proceed as follows: let $d > 0$ be the infimum of the distance of two hyperplanes that are parallel and such that Ω is contained into the strip limited by them. As the equation (9) is equivariant with respect to rotations and translations in $\mathbb{R}^N$, we can assume that

$$\Omega \subset \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : 0 < x_1 < d\}. \tag{43}$$

Let ζ be the unique positive solution of

$$\begin{aligned} -\varepsilon(|\zeta_{x_1}|^{p-2}\zeta_{x_1})_{x_1} + f(\zeta) = 0 \quad &\text{in } (0, d), \\ \zeta(0) = \zeta(d) = 0. \end{aligned} \tag{44}$$

It is clear that $\tilde{\zeta}(x) = \zeta(x_1)$ satisfies

$$\begin{aligned} -\varepsilon\Delta_p\tilde{\zeta} + f(\tilde{\zeta}) = 0 \quad &\text{in } \Omega, \\ \tilde{\zeta} \geq 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{45}$$

As before there exists a solution $\tilde{u}$ such that for some $\alpha < 1$

$$\alpha u \leq \tilde{u} \leq \zeta \tag{46}$$

and by uniqueness $\tilde{u} = u \leq \zeta$. If $0 < \zeta < 1$ in $(0, d)$ we deduce that the coincidence set Ω_ε is empty. In the particular case of equation (1) the coincidence set is empty if

$$\begin{aligned} &\lambda^{1/p} d^{(q-1)/(q+1-p)} \\ &< 2\int_0^1 \left(\frac{q+1-p}{(p-1)(q+1)} + \frac{p}{(p-1)(q+1)}\sigma^{q+1} - \frac{\sigma^p}{p-1}\right)^{-1/p} d\sigma \end{aligned} \tag{47}$$

[GV, Remark 2.3].

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