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ON PRIME IDEALS IN RINGS OF SEMIALGEBRAIC FUNCTIONS

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ABSTRACT. It is proved that if $\mathfrak{p}$ is a prime ideal in the ring $S(M)$ of semialgebraic functions on a semialgebraic set M, the quotient field of $S(M)/\mathfrak{p}$ is real closed. We also prove that in the case where M is locally closed, the rings $S(M)$ and $P(M)$—polynomial functions on M—have the same Krull dimension. The proofs do not use the theory of real spectra.

Let R be a real closed field. The only topology we consider in R^n is the euclidean topology. For every semialgebraic subset $M \subset R^n$ we denote by $S(M)$ the ring of semialgebraic functions on M, i.e., continuous maps $f: M \to R$ whose graph is a semialgebraic subset of R^{n+1}, and $P(M)$ stands for the ring of polynomial functions on M, i.e., restrictions to M of polynomials in $R[x_1, \dots, x_n]$. In this note we give elementary proofs of the following results.

Theorem 1. *For every prime ideal $\mathfrak{p}$ of $S(M)$, the quotient field of $S(M)/\mathfrak{p}$ is real closed.*

Theorem 2. *Let π: $\operatorname{Spec} S(M) \to \operatorname{Spec} P(M)$ be the map induced by the inclusion $P(M) \hookrightarrow S(M)$, and assume that M is locally closed. Then*:

(1) *The fibers of π are T_1-spaces.*
(2) *The Krull dimensions of $P(M)$ and $S(M)$ are equal.*
(3) *If $M \subset R^n$ is algebraic, the image of π is the set $\operatorname{Spec}^r(M)$ of real prime ideals of $P(M)$.*

For every ring A, $\operatorname{Spec} A$ denotes the prime spectrum of A endowed with its Zariski topology. The ideal $\mathfrak{a}$ in A is real if whenever $f_1, \dots, f_k \in A$ and $f_1^2 + \dots + f_k^2 \in \mathfrak{a}$, then $f_1, \dots, f_k \in \mathfrak{a}$. Theorem 1 can be deduced from [9, Corollary 3.26, §1, and Theorem 1.1]. Schwartz's proof involves his theory of real closed spaces. This can be viewed as the semialgebraic counterpart of results by Henriksen and Isbell [4] and Isbell [5]. In fact, if $\mathfrak{m}$ is a maximal ideal in the ring of real-valued continuous functions on a normal topological space X, it is proved in [4] that the quotient $C(X)/\mathfrak{m}$ is a real closed field. In [5], the normality condition on X is dropped. The second part of Theorem 2

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was proved, using the theory of real spectra by Carral and Coste [2] for locally closed semialgebraic sets M, and by Gamboa and Ruiz [3, Proposition 1.4] for arbitrary semialgebraic M.

Let us fix some notation. For each ideal $\mathfrak{a}$ in $S(M)$, the set of common zeros in M of functions in $\mathfrak{a}$ is denoted by $Z(\mathfrak{a})$. If $\mathfrak{a} = f \cdot S(M)$ for some $f \in S(M)$ we abbreviate $Z(f) = Z(f \cdot S(M))$. Given a subset $X \subset M$, the ideal of all functions $f \in S(M)$ such that $X \subset Z(f)$ is denoted by $\mathscr{I}(X)$, while $\overline{X}^z$ is the smallest algebraic subset of R^n containing X. Finally, for each $f \in S(M)$ we denote by $|f| \in S(M)$ the "absolute value of f".

Proof of Theorem 1. Let $A = S(M)/\mathfrak{p}$ and let $E = q \cdot f(A)$ be its quotient field. First we must prove that either x or $-x$ is a square in E for every $x \in E$. Write $x = (f+\mathfrak{p})(g+\mathfrak{p})^{-1}$, $f, g \in S(M)$ and $g \notin \mathfrak{p}$. If $h = fg$ we get $(|h| - h)(|h| + h) = 0 \in \mathfrak{p}$, and so either $|h| - h \in \mathfrak{p}$ or $|h| + h \in \mathfrak{p}$. In the first case, $x = (h+\mathfrak{p})(g+\mathfrak{p})^{-2} = (|h|+\mathfrak{p})(g+\mathfrak{p})^{-2}$ is a square in E, since $|h|$ is a square in $S(M)$. Analogously, $-x$ is a square in E in the second case. Hence, we must only show that each odd degree polynomial $P \in E[T]$ has at least one root in E. We may suppose that $P \in A[T]$ is monic. In fact, if we write $P(T) = (a_0 T^m + a_1 T^m + \cdots + a_m) \cdot b^{-1}$, $a_i \in A$, $a_0, b \in A \backslash \{0\}$, then we construct the monic polynomial $Q(T) = T^m + \sum_{j=1}^m a_j \cdot a_0^{j-1} T^{m-j} \in A[T]$, and if $\alpha \in E$ is a root of Q, then $\alpha \cdot a_0^{-1} \in E$ is a root of P. Thus, from now on we put $P(T) = T^m + \sum_{j=1}^m (f_j + \mathfrak{p}) T^{m-j}$, $f_j \in S(M)$, and m is odd. Let us consider the polynomial $F_0 = T^m + \sum_{j=1}^m x_j T^{m-j} \in R[x_1, \dots, x_m, T]$. Clearly, F_0 and its derivatives $F_j = \partial F_0 / \partial T^j$, $1 \le j \le m$, are monic (modulo factors in $\mathbb{N}$) with respect to T, and the family $\mathscr{F} = \{F_0, F_1, \dots, F_m\}$ is stable under derivation (with respect to T). Therefore, if $(A_i; \zeta_{ij} : 1 \le i \le k,\ 1 \le j \le k(i))$ is a "saucissonage" of $\mathscr{F}$ (see [1, 2.3.4], then functions $\zeta_{ij} \in S(A_1)$ can be extended to the closure $\overline{A}_i$ by [1, 2.5.6] and so, using the semialgebraic Tietze's extension theorem [1, 2.6.10], there exist semialgebraic functions $\eta_{ij} \in S(R^m)$ such that η_{ij} restricted to A_i coincides with ζ_{ij}. By the very definition of "saucissonage", and since m is odd, there exists for every $1 \le i \le k$ an index $l(i) \in \{1, \dots, k(i)\}$ such that $F_0(x, \eta_{i,l(i)}(x)) = 0$ for each point $x \in A_i$.

On the other hand, if $\varphi = (f_1, \dots, f_m) : M \to R^m$, the compositum $g_i = \eta_{i,l(i)} \circ \varphi$ belong to $S(M)$, and all reduce to proving that $g_i + \mathfrak{p}$ is a root of P for some i. For every point $y \in M$, $x = \varphi(y) \in R^m = A_1 \cup \cdots \cup A_k$ and so

$$\prod_{j=1}^{k} F_0(\varphi(y), g_j(y)) = \prod_{i=1}^{k} F_0(x, \eta_{i,l(i)}(x)) = 0.$$

Consequently, the polynomial $H(T) = T^m + \sum_{j=1}^m f_j \cdot T^{m-1}$ verifies that the product $H(g_1) \cdots H(g_k) = 0$, since for each $y \in M$,

$$F_0(\varphi(y), g_i(y)) = g_i(y)^m + \sum_{j=1}^{m} f_j(y) g_i^{m-j}(y).$$

Finally, since $\mathfrak{p}$ is prime, $H(g_i) \in \mathfrak{p}$ for some $1 \le i \le k$, i.e., $P(g_i + \mathfrak{p}) = 0$.

Proof of Theorem 2. (1) We must prove that $\mathfrak{a} \cap P(M) \subsetneq \mathfrak{b} \cap P(M)$ for given prime ideals $\mathfrak{a} \subsetneq \mathfrak{b}$ in $S(M)$. Let us take $f \in \mathfrak{b} \backslash \mathfrak{a}$. Its zero-set $Z(f)$ is a closed semialgebraic subset of M and so, by the finiteness theorem (see [6] or [1, 2.7.1]), there exist polynomial functions $f_{ij} \in P(M)$ such that $Z(f) = \bigcup_{i=1}^{m}\{x \in M : f_{i1}(x) \geq 0, \dots, f_{ik}(x) \geq 0\}$. Define $h_i = \sum_{j=1}^{k}(f_{ij} - |f_{ij}|)^2$, $i = 1, \dots, m$. Then $Z(f) = Z(h)$ for $h = h_1 \cdots h_m$, and so $\mathcal{I}(Z(f)) = \mathcal{I}(Z(h))$. Hence, by the semialgebraic Nullstellensatz [1, 2.6.7], $\sqrt{f \cdot S(M)} = \sqrt{h \cdot S(M)}$. In particular, since $f \notin \mathfrak{a}$, we conclude that every $h_i \notin \mathfrak{a}$ and, from $f \in \mathfrak{b}$, also $h \in \mathfrak{b}$ and so $h_i \in \mathfrak{b}$ for some $1 \leq i \leq m$. From Theorem 1, the quotient field of $S(M)/\mathfrak{b}$ is formally real, i.e., $\mathfrak{b}$ is a real ideal, and $h_i \in \mathfrak{b}$ for some $1 \leq i \leq m$. Thus, $f_{ij} - |f_{ij}| \in \mathfrak{b}$ for all $1 \leq j \leq k$, and since $h_i \notin \mathfrak{a}$, there exists j with $f_{ij} - |f_{ij}| \in \mathfrak{b}\backslash\mathfrak{a}$. To finish we shall check that $g = f_{ij} \in [\mathfrak{b} \cap P(M)]\backslash[\mathfrak{a} \cap P(M)]$. In fact, $0 = (g - |g|) \cdot (g + |g|)$ and since $\mathfrak{a}$ is prime, $g + |g| \in \mathfrak{a} \subset \mathfrak{b}$. Thus, $g = [(g - |g|) + (g + |g|)] \cdot 2^{-1} \in \mathfrak{b}$.

On the other hand, if $g \in \mathfrak{a}$, then $|g|^2 = g^2 \in \mathfrak{a}$ also, i.e., $|g| \in \mathfrak{a}$, which implies $g - |g| \in \mathfrak{a}$, absurd.

(2) The inequality $\dim S(M) \leq \dim P(M)$ is a consequence of part (1). Let $d = \dim P(M) = \dim \overline{M}^{z} = \dim M$. Then M contains a closed semialgebraic subset K semialgebraically homeomorphic to the cube $I = [-1, 1]^d \subset R^d$. From Tietze's theorem [1, 2.6.10], $\dim S(M) \geq \dim S(K) = \dim S(I)$ and so it suffices to see that $d \leq \dim S(I)$. In the polynomial ring $A = R[x_1, \dots, x_d]$ we consider the chain of prime ideals

$$(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d; \qquad \mathfrak{p}_k = (x_1, \dots, x_k) \cdot A.$$

The quotient fields $E_k = q \cdot f(A/\mathfrak{p}_k) \approx R(x_{k+1}, \dots, x_d)$ are formally real, and each ordering in E_k can be extended to E_{k-1}. So we can choose cones α_k of nonnegative elements in E_k such that (E_k, α_k) is an ordered extension of (E_{k+1}, α_{k+1}). Now define the ideals

$$\begin{aligned}\mathfrak{q}_k = \{f \in S(I) : {}& \text{there exists } g_1, \dots, g_l \in A \text{ such that } g_i + \mathfrak{p}_k \in \alpha_k \text{ and} \\ & P(g_1, \dots, g_l) = \{x \in I : g_1(x) \geq 0, \dots, g_l(x) \geq 0\} \subset Z(f)\}.\end{aligned}$$

Obviously $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_d$ and so it is enough to check that $\mathfrak{q}_k$ is prime and $\mathfrak{q}_k \cap A = \mathfrak{p}_k$. In what follows $\overline{g}$ denotes the class $\bmod\, \mathfrak{p}_k$ of $g \in A$. Let $f, g \in S(I)$ such that $fh \in \mathfrak{q}_k$. Then $Z(f) \cup Z(h)$ contains the set $P(g_1, \dots, g_r)$ for some $g_1, \dots, g_r \in A$ with $\overline{g}_i \in \alpha_k$. From the finiteness theorem [6] we can write $Z(f) = \bigcup_{i=1}^{m} P(f_{i1}, \dots, f_{il})$, $Z(h) = \bigcup_{i=1}^{m} P(h_{i1}, \dots, h_{il})$ for certain $f_{ij}, h_{ij} \in A$ and in case neither f nor h belong to q_k, there exists a family $\{f_{i,j(i)}, h_{i,l(i)} : 1 \leq i \leq m\}$ such that $\overline{f_{ij(i)}} \in \alpha_k$, $\overline{h_{ij(i)}} \notin \alpha_k$. From Artin-Lang theorem [1, 4.1.2] there exists a homomorphism $\mathcal{S}\colon A/\mathfrak{p}_k \to R$ such that:

$$\begin{array}{ll}\text{(i) } \mathcal{S}(\overline{g}_s) \geq 0; & \text{(ii) } \mathcal{S}(\overline{f_{ij(i)}}) < 0; \\ \text{(iii) } \mathcal{S}(\overline{h_{ij(i)}}) < 0; & \text{(iv) } p = (\mathcal{S}(\overline{x}_1), \dots, \mathcal{S}(\overline{x}_d)) \in I.\end{array}$$

Then, each $g_s(p) = \mathcal{S}(\overline{g}_s) \geq 0$ and so $p \in Z(f) \cup Z(h)$ which is false since $f_{ij(i)}(p) = \mathcal{S}(\overline{f_{ij(i)}}) < 0$ and $h_{il(i)}(p) < 0$ for all i. Hence $\mathfrak{q}_k$ is prime. Also, for $f \in \mathfrak{p}_k$ we have $Z(f) = P(f, -f)$ and $\overline{f}, -\overline{f} \in \alpha_k$, and so $\mathfrak{p}_k \subset \mathfrak{q}_k \cap A$. Finally, if some $f \in \mathfrak{q}_k \cap A$ exists, but $f \notin \mathfrak{p}_k$, then $Z(f) \supset P(g_1, \dots, g_r)$ for some $g_i \in A$ with $\overline{g}_i \in \alpha_k$. Again from the Artin-Lang theorem we get a homomorphism $\psi\colon A/\mathfrak{p}_k \to R$ such that $\psi(\overline{f}) \neq 0$, $\psi(\overline{g}_s) \geq 0$, and $q = (\psi(\overline{x}_1), \dots, \psi(\overline{x}_d)) \in I$, i.e., $q \in P(g_1, \dots, g_r) \setminus Z(f)$, which is absurd.

(3) Each prime ideal in $S(M)$ is real. Hence $\operatorname{Spec}^r P(M)$ contains the image of π. For the converse, assume first that M is irreducible and $\mathfrak{p} = \mathfrak{p}_0$ is the zero ideal in $P(M)$. Let $a \in M$ be a regular point of dimension $d = \dim M$ of M, and let U be an open neighborhood of a in R^n such that there exists a semialgebraic homeomorphism $F: \Delta_d = [-1, 1]^d \to M \cap U$ with $F(0) = a$. For every $\varepsilon \in R^+$ let us denote $A_\varepsilon = \{x \in \Delta_{d-1} : 0 < x_i < \varepsilon,\ i = 1, \dots, d-1\}$. For every semialgebraic function $\mathscr{S}: \overline{\Delta}_\varepsilon \to R^+ \cup \{0\}$, define

$$A_\varepsilon(\mathscr{S}) = \{(x', x_d) \in R^d : x' \in \Delta_\varepsilon \text{ and } 0 < x_d < \mathscr{S}(x')\}.$$

Then we construct a prime ideal in $S(M)$ as follows: $\mathfrak{q} = \{h \in S(M):$ there exists $\varepsilon \in R^+$ and a semialgebraic function $\mathscr{S}: \overline{\Delta}_\varepsilon \to R^+ \cup \{0\}$ with $Z(\mathscr{S}) = \{a\}$ such that $(f|M \cap U) \circ F$ vanishes on $A_\varepsilon\}$. Moreover, $\mathfrak{q} \cap P(M) = \mathfrak{p}_0$ since every $f \in \mathfrak{q} \cap P(M)$ vanishes on $M \cap U$ and so over $\overline{M \cap U}^z = M$.

If M is irreducible and $\mathfrak{p}$ is an arbitrary prime ideal in $P(M)$, the zero set $N = Z(\mathfrak{p}) \subset M$ is an irreducible algebraic set and so there exists a prime ideal $\mathfrak{q}_N$ of $S(N)$ lying over the zero ideal of $P(N)$. Let $r^*: \operatorname{Spec} S(N) \to \operatorname{Spec} S(M)$ be the map induced by the restriction homomorphism $r: S(M) \to S(N)$. Then $\mathfrak{q} = r^*(\mathfrak{q}_N)$ verifies $\mathfrak{q} \cap P(M) = \mathfrak{p}$, by the real Nullstellensatz [1, 4.4.3].

Finally let M be arbitrary with irreducible components $M_1, \dots, M_k$ and let $\mathfrak{p}$ be a real prime ideal in $P(M)$. Write $A = R[x_1, \dots, x_n]$ and I (resp. I_i) the ideal of polynomials in A vanishing on M (resp. M_i). There exists a prime ideal $\mathfrak{p}^*$ in A containing $I = I_1 \cap \cdots \cap I_k$ such that $\mathfrak{p} = \mathfrak{p}^*/I$. We can suppose that $\mathfrak{p}^*$ contains I_1 and so $\mathfrak{p}_1 = \mathfrak{p}^*/I_1$ is a real prime ideal in $P(M_1)$. Hence there exists a prime ideal $\mathfrak{q}_1$ in $S(M_1)$ such that $\mathfrak{q}_1 \cap P(M_1) = \mathfrak{p}_1$ and so, if $r_1: S(M) \to S(M_1)$ is the restriction homomorphism, we get $\mathfrak{q} = r_1^*(\mathfrak{q}_1) \in \operatorname{Spec} S(M)$ such that $\mathfrak{q} \cap P(M) = \mathfrak{p}$.

Remark. Part (3) of Theorem 2 is no longer true for more general semialgebraic subsets $M \subset R^n$. Consider for example a nonfinite semialgebraic subset M of R, $M \neq R$, a point $a \in R \setminus M$, and the function $f : M \to R$ defined by $f(x) = x - a$. Then $\mathfrak{a} = f \cdot P(M)$ is a real ideal but since $Z(f)$ is empty there is no prime ideal in $S(M)$ lying over $\mathfrak{p}$.

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