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COMPLEMENTED SUBSPACES AND AMENABILITY: A COUNTEREXAMPLE

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Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday

ABSTRACT. We give an example of a left amenable discrete semigroup S such that $l^\infty(S)$ has weak*-closed selfadjoint left translation invariant subalgebras that are weak*-complemented but not invariantly complemented in $l^\infty(S)$. This resolves negatively a problem raised by Lau.

Let S be a (discrete) semigroup and let $l^\infty(S)$ be the Banach algebra of bounded complex-valued functions on S with pointwise operations and supremum norm. Following [2] we say that a weak*-closed left translation invariant subspace X of $l^\infty(S)$ is *invariantly complemented* in $l^\infty(S)$ if X admits a left translation invariant closed complement, or equivalently, X is the range of a continuous projection on $l^\infty(S)$ commuting with left translations. In [3, Problem 3] Lau asked if left amenability of S implies each of the following properties:

(C) Each weak*-closed left translation invariant complemented subspace of $l^\infty(S)$ is invariantly complemented in $l^\infty(S)$.
(C*) Each weak*-closed selfadjoint left translation invariant subalgebra of $l^\infty(S)$ is invariantly complemented in $l^\infty(S)$.

Notice that Lau's argument in the proof of Theorem 3.3 in [2] shows that each of (C) and (C*) implies left amenability of S. We also note that each of (C) and (C*) is equivalent to left amenability of S when S is a group (see [2, Theorem 3.3; 4, Theorem 1]). The purpose of the present note is to give an example that shows the answer to Lau's problem is negative. Our example also solves a question in [2, p. 232] negatively.

Recall that a weak*-closed subspace X of $l^\infty(S)$ is called *weak*-complemented* in $l^\infty(S)$ if there exists a weak*-weak* continuous projection from $l^\infty(S)$ onto X. Our example is a direct consequence of the following result.

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Theorem. *Let S be a semigroup and let E be a subset of S with the following properties*:

(a) E^{c} (*the complement of E in S*) *is a left ideal of S*.
(b) *There exists $t \in S$ such that $tS \cap E^{\mathrm{c}} \cap (tE^{\mathrm{c}})^{\mathrm{c}}$ is nonempty.*

Let X_E be the set of all functions of $l^\infty(S)$ that vanish on E^{c}. Then X_E is a weak-closed selfadjoint left translation invariant subalgebra of $l^\infty(S)$ that is weak*-complemented but not invariantly complemented in $l^\infty(S)$.*

Proof. Obviously X_E is a weak*-closed selfadjoint subalgebra of $l^\infty(S)$. It also follows immediately from the assumption (a) on E and the definition of X_E that X_E is left translation invariant. For each $f \in l^\infty(S)$, define $Pf \in l^\infty(S)$ as follows: $Pf(s) = f(s)$ if $s \in E$ and $Pf(s) = 0$ if $s \in E^{\mathrm{c}}$. Then P is a continuous projection from $l^\infty(S)$ onto X_E. Furthermore, P is weak*-weak* continuous because P is the conjugate operator of the continuous linear operator Q on $l^1(S)$ (the Banach space of all absolutely summable complex-valued functions on S) obtained by the same method as that P was defined. Thus X_E is weak*-complemented in $l^\infty(S)$. It remains to show that X_E is not invariantly complemented in $l^\infty(S)$. To see this, suppose that X_E is invariantly complemented in $l^\infty(S)$ and denote by Y a left translation invariant closed complement. Our assumption (b) implies that there exist t and u in S such that $tu \in E^{\mathrm{c}} \cap (tE^{\mathrm{c}})^{\mathrm{c}}$. Now consider $f = \chi_{\{tu\}}$ (the characteristic function of one element set $\{tu\}$) and represent f as

$$f = g + h \qquad (g \in X_E \text{ and } h \in Y).$$

If $s \in E$ then $s \neq tu$ because $tu \in E^{\mathrm{c}}$. Thus $f(s) = 0$ and, therefore, $h(s) = -g(s)$. Let $s \in E^{\mathrm{c}}$ and $s \neq tu$. Then $f(s) = 0$. Since $g \in X_E$, it follows that $g(s) = 0$. Hence we have $h(s) = 0$. If $s \in E^{\mathrm{c}}$ and $s = tu$, then $f(s) = 1$ and $g(s) = 0$, and so $h(s) = 1$. Thus we have

$$h(s) = \begin{cases} 0 & \text{if } s \in E^{\mathrm{c}} \text{ and } s \neq tu, \\ 1 & \text{if } s \in E^{\mathrm{c}} \text{ and } s = tu, \\ -g(s) & \text{otherwise.} \end{cases}$$

Let $s \in E^{\mathrm{c}}$. Then $ts \in E^{\mathrm{c}}$. Since $tu \in (tE^{\mathrm{c}})^{\mathrm{c}}$, it follows that $ts \neq tu$. Thus we have $L_t h(s) = 0$ for each $s \in E^{\mathrm{c}}$. ($L_t h$ denotes the left translate of h by t.) Hence $L_t h$ is an element of X_E. $L_t h$ is also contained in Y because Y is left translation invariant. Consequently $L_t h$ must be equal to the constant function 0. But, since $L_t h(u) = h(tu) = 1$, we obtain a contradiction. This completes the proof.

Let us now give a typical example of a left amenable semigroup satisfying the assumption of our theorem.

Example. Let S be the direct product of the additive semigroup $\mathbf{N}$ of positive integers and a left amenable semigroup T. Since $\mathbf{N}$ is abelian, $\mathbf{N}$ is amenable and hence S is also amenable [1, Theorems (17.5) and (17.18)]. For $n \in \mathbf{N}$, let

$$E(n) = \{(k, t) \in S : k \in \mathbf{N} \text{ with } k \leq n \text{ and } t \in T\}.$$

Then $E(n)$ satisfies the assumption of our theorem for each n. Indeed, it is clear that $E(n)^{c}$ is a left ideal of S. Now choose and fix arbitrary elements t and u in T. Put $x = (1, t)$ and $y = (n, u)$. Then

$$xy = (1 + n, tu) \in E(n)^{c} \cap E(n + 1).$$

Since $E(n + 1)$ is contained in $(xE(n)^{c})^{c}$, it follows that $xy \in (xE(n)^{c})^{c}$. Thus we have

$$xy \in xS \cap E(n)^{c} \cap (xE(n)^{c})^{c},$$

and hence $E(n)$ satisfies assumption (b) of the Theorem. Consequently, for each n,

$$X_{E(n)} = \{f \in l^{\infty}(S) : f = 0 \text{ off } E(n)\}$$

is a weak*-closed selfadjoint left translation invariant subalgebra of $l^{\infty}(S)$ that is weak*-complemented but not invariantly complemented in $l^{\infty}(S)$. This resolves Lau's problem [3, Problem 3] negatively. We also see that the answer to Lau's problem is negative even when S is abelian.

Remark. In [2, p. 232] Lau raised the question: If S is a left amenable semigroup and if X is a weak*-complemented left translation invariant weak*-closed subspace of $l^{\infty}(S)$, then is X invariantly complemented? Notice that the answer is affirmative when S is a group (see [2, Corollary 4.2]). Our example gives a negative answer to this question.

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