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SYMMETRIC FUNCTIONS, LEBESGUE MEASURABILITY, AND THE BAIRE PROPERTY

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ABSTRACT. In this paper, we generalize some results of Stein and Zygmund and of Evans and Larson concerning symmetric functions. In particular, we show that if f is Lebesgue measurable or has the Baire property in the wide sense, then the set of symmetric points of f is Lebesgue measurable or has the Baire property in the wide sense, respectively. We also give some examples that show that these results cannot be improved in a certain sense. Finally, we show that there are plenty of examples of functions that are both Lebesgue measurable and have the Baire property in the wide sense, yet the set of points where each of the functions is symmetric and discontinuous has the same cardinality as that of the continuum.

I. INTRODUCTION

A function $f: R \to R$ is said to be symmetric at $x \in R$ if

$$\lim_{h\to 0} f(x+h) + f(x-h) - 2f(x) = 0,$$

and f is symmetric if f is symmetric at every $x \in R$.

In 1964 Stein and Zygmund in [8] showed that if $f: R \to R$ is Lebesgue measurable and f is symmetric on a Lebesgue measurable set M, then f is continuous a.e. on M. In 1984 Evans and Larson in [3] showed that if $f : R \to R$ has the Baire property in the wide sense and f is symmetric on a set M which has the Baire property in the wide sense, then the set of all points of M where f is discontinuous is of first category.

In this paper, we show that in the aforementioned theorems the additional hypothesis of M being measurable or having the Baire property in the wide sense is not necessary. Denote by $S(f)$ the set of points where f is symmetric and by $C(f)$ the set of points where f is continuous. We show that if f is Lebesgue measurable or has the Baire property in the wide sense, then $S(f)\backslash C(f)$ is of measure zero or first category, respectively. As corollaries, we

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get that if f is Lebesgue measurable or has the Baire property in the wide sense, then $S(f)$ is Lebesgue measurable or has the Baire property in the wide sense, respectively.

In 1964 Neugebauer in [5] showed that if $f: R \to R$ is symmetric and Lebesgue measurable, then f has to be of Baire class 1. In 1984 Evans and Larson in [3] showed that if $f: R \to R$ is symmetric and has the Baire property in the wide sense, then f is of Baire class 1. It follows from the Evans-Larson theorem and the Neugebauer theorem that a symmetric function $f: R \to R$ is Lebesgue measurable iff it has the Baire property in the wide sense; however, this is not true if the function is not symmetric on the entire line as will be evident by Examples 8 and 9.

Lastly, we show that there are plenty of examples of functions which are simultaneously Lebesgue measurable and have the Baire property in the wide sense, yet the set of points where each of the functions is symmetric and discontinuous is uncountable.

Let us now state some definitions and background theorems.

Recall that a set $M \subseteq R$ has the Baire property in the wide sense, or for short $M \in B_w$, if it is the difference of an open set and a first category set. This is equivalent to saying that M is the union of two sets, one of which is G_δ and the other of the first category. A set M is categorically dense in some interval I means that $M \cap J$ is second category for every interval $J \subseteq I$. Also recall that every second category set is categorically dense in some interval. A set $M \subseteq R$ has the Baire property in the restricted sense, or for short $M \in B_r$, means that M has the Baire property in the wide sense relative to every perfect set P (i.e., $M \cap P$ is B_w relative to P). A set $M \subseteq R$ is always first category, or for short $M \in \mathrm{AFC}$, if every subset of M has the Baire property in the restricted sense. A function $f: R \to R$ has the Baire property in the wide sense (the Baire property in the restricted sense) means that the preimage of every open set under f has the Baire property in the wide sense (the Baire property in the restricted sense).

We will let $\mu(M)$, $\mu^*(M)$, and $\mu_*(M)$ denote the Lebesgue measure of M, the outer Lebesgue measure of M, and the inner Lebesgue measure of M, respectively. Also recall that saying M is Lebesgue measurable is equivalent to saying that for every perfect set P with positive measure, there exists a perfect set $Q \subset P$ with positive measure such that $Q \subseteq M$ or $Q \subseteq M^c$. From this definition, it easily follows that if M is a set with positive outer measure then there exists a perfect set P with positive measure such that M has full outer measure in P. A set $M \subseteq R$ is universally measurable, or for short $M \in U$, iff M is measurable according to the completion of every nonatomic Borel measure on R. A set $M \subseteq R$ is U_0 iff every subset of M is universally measurable. A function $f: R \to R$ is universally measurable means that the preimage of every open set under f is universally measurable.

An uncountable set $M \subseteq R$ is a Lusin set (Sierpinski set) iff the intersection of M with every first category set (measure zero set) is countable. Under the assumption of the continuum hypothesis, there are uncountable Lusin and Sierpinski sets. Also recall that the Lusin sets do not have the Baire property in the wide sense, yet they are U_0; the Sierpinski sets are nonmeasurable, yet they are AFC. Refer to [4] for more information about the Baire property, universal measurability, AFC sets, U_0 sets, Lusin sets, and Sierpinski sets.

If $r \in R$ and each of A and B is a subset of the real line, then $rA = \{rx : x \in A\}$, $r + A = \{r + x : x \in A\}$, and $A + B = \{x + y : x \in A \text{ and } y \in B\}$.

II. Main results

Theorem 1. *If $f : R \to R$ has the Baire property in the wide sense, then $S(f)\backslash C(f)$ is of first category.*

Proof. Suppose that $S(f)\backslash C(f)$ is second category. Then $S(f)\backslash C(f)$ is categorically dense in some closed interval I. We shall show that this situation is impossible by showing that $C(f)$ is actually dense in this I.

Suppose that $C(f)$ is not dense in I and I_1 is a closed subinterval of I containing no point of $C(f)$. Using the fact that $I_1 \cap C(f) = \varnothing$ and the Baire category theorem, we may obtain a closed subinterval $I_2 \subset I_1$ and a rational $\varepsilon > 0$ such that $\operatorname{osc}(f, x) > \varepsilon$ for each $x \in I_2$. Here $\operatorname{osc}(f, x)$ denotes the oscillation of f at x.

Now let $\{J_k\}$ be an enumeration of all open intervals of length $\varepsilon/3$ having rational end points. There must be k such that $f^{-1}(J_k) \cap I_2$ is second category. Since f has the Baire property in the wide sense, $f^{-1}(J_k) \cap I_2$ contains a second category G_δ set, which, consequently, is residual in some closed interval $I_3 \subseteq I_2$.

Let $J \equiv J_k \equiv (a, b)$ and $K \equiv R\backslash(a - 2\varepsilon/3, b + 2\varepsilon/3)$. Since $\operatorname{osc}(f, x) > \varepsilon$ for each $x \in I_3$, $f^{-1}(K)$ is dense in I_3. We now make the following claim: For each closed interval $H \equiv [u, v] \subseteq I_3$, there is a closed subinterval $H^* \subseteq H$ such that $\{(s + t)/2 : s \in f^{-1}(J) \cap H \text{ and } t \in f^{-1}(K) \cap H\}$ contains a dense G_δ subset of H^*. To see this, let $p \in f^{-1}(K) \cap H$. Since $f^{-1}(J) \cap H$ contains a dense G_δ subset of H, it follows that the set $\{(s + p)/2 : s \in f^{-1}(J) \cap H\}$ contains a G_δ subset which is dense in $H^* = ((u + p)/2, (p + v)/2)$.

For each natural number j, let $G_j = \{x \in I_3 :$ there is a $0 < h < 1/j$ such that one of $x + h$, $x - h$ belongs to $f^{-1}(K)$ while the other belongs to $f^{-1}(J)\}$. Each G_j is residual in I_3. To see this, suppose that $I_3\backslash G_j$ is second category and let $H \subseteq I_3$ be a closed interval of length less than $1/j$ in which $I_3\backslash G_j$ is categorically dense. Applying the above claim to this interval H yields a contradiction.

Now, since each G_j is residual in I_3, so is $G \equiv \bigcap_{j=1}^{\infty} G_j$. Let $x \in S(f) \cap f^{-1}(J) \cap G$, which is nonempty since G and $f^{-1}(J)$ are residual in I_3 and $S(f)$ is second category in I_3. We have that for each j, there is a $0 < h_j < 1/j$ such that one of $x + h_j$, $x - h_j$ belongs to $f^{-1}(K)$ while the other belongs to $f^{-1}(J)$. Without loss of generality, suppose that $x + h_j \in f^{-1}(K)$. Then

$$\begin{aligned} |f(x + h_j) + f(x - h_j) - 2f(x)| &\geq |f(x + h_j) - f(x)| - |f(x - h_j) - f(x)| \\ &> 2\varepsilon/3 - \varepsilon/3 = \varepsilon/3, \end{aligned}$$

which contradicts the assumption that $x \in S(f)$ and completes the proof.

Corollary 2. *If $f : R \to R$ has the Baire property in the wide sense, then $S(f)$ has the Baire property in the wide sense.*

Theorem 3. *If $f : R \to R$ is Lebesgue measurable, then $S(f)\backslash C(f)$ has measure zero.*

Proof. Suppose that $S(f)\backslash C(f)$ has positive outer measure. Then there exists a perfect set P with positive measure such that $S(f)\backslash C(f)$ has full measure in

P. We shall show that this situation is impossible by showing that $C(f)$ has full measure in this P.

Suppose that $C(f)$ does not have full measure in P and P_1 is a perfect subset of P such that $\mu(P_1) > 0$ and $P_1 \cap C(f) = \varnothing$. Since $P_1 \cap C(f) = \varnothing$, there exists a perfect set $P_2 \subseteq P_1$ and a rational number $\varepsilon > 0$ such that $\mu(P_2) > 0$ and $\operatorname{osc}(f, x) > \varepsilon$ for each $x \in P_2$.

Now let $\{J_k\}$ be an enumeration of all open intervals of length $\varepsilon/3$ having rational end points. Since f is Lebesgue measurable, there exists k such that $f^{-1}(J_k) \cap P_2$ has positive measure. Let $P_3 \subseteq P_2$ be a bounded perfect set such that $P_3 \subseteq f^{-1}(J_k)$, and let $\mu(P_3) > 3(d-c)/4$, where c and d are the left end point and the right end point of P_3, respectively.

Let $J \equiv J_k \equiv (a, b)$ and $K \equiv R\backslash(a - 2\varepsilon/3, b + 2\varepsilon/3)$. For each natural number j, let $G_j = \{x \in P_3\}$: there is a $0 < h < 1/j$ such that one of $x + h$, $x - h$ belongs to $f^{-1}(K)$ while the other belongs to $f^{-1}(J)\}$. Each G_j has inner measure at least $3(d-c)/8$. To see this, consider the following argument: Let $\{I_t\}_{j=1}^{n}$ be a partition of $[c, d]$ such that mesh of $\{I_t\}_{j=1}^{n} < 1/j$. Let $A \equiv \{1 \le m \le n$: the interior of I_m contains a point of $P_3\}$. For each $m \in A$, let $q_m \in f^{-1}(K) \cap I_m$. We know that there are such q_m because $\operatorname{osc}(f, x) > \varepsilon$ for each $x \in P_3$. Then $(q_m + (P_3 \cap I_m))/2 \subseteq G_j \cap I_m$ and

$$\begin{aligned}\tfrac{3}{8}(d-c) \le \mu(P_3)/2 &= \sum_{m\in A} \mu(P_3 \cap I_m)/2 \\ &= \sum_{m\in A} \mu(q_m + (P_3 \cap I_m))/2 \\ &\le \sum_{m\in A} \mu_*(G_j \cap I_m) \le \mu_*(G_j).\end{aligned}$$

Now let $G = \bigcap G_j$. Since for each j, $\mu_*(G_j) > 3(d-c)/8$, we have that $\mu_*(G) > 3(d-c)/8$. $G \cap P_3 \cap S(f) \neq \varnothing$ follows from the fact that $G \cap P_3$ has positive inner measure and $S(f)$ has full outer measure in P and hence in P_3.

Let $x \in G \cap P_3 \cap S(f)$. We have that for each j, there is a $0 < h_j < 1/j$ such that one of $x + h_j$, $x - h_j$ belongs to $f^{-1}(K)$ while the other belongs to $f^{-1}(J)$. Without loss of generality, suppose that $x + h_j \in f^{-1}(K)$. Then

$$\begin{aligned}|f(x+h_j) + f(x-h_j) - 2f(x)| &\ge |f(x+h_j) - f(x)| - |f(x-h_j) - f(x)| \\ &> 2\varepsilon/3 - \varepsilon/3 = \varepsilon/3,\end{aligned}$$

which contradicts the assumption that $x \in S(f)$ and completes the proof.

Corollary 4. *If f is Lebesgue measurable, then $S(f)$ is Lebesgue measurable.*

In [2], under the assumption of the continuum hypothesis, Erdös constructed two groups, one of which was simultaneously measure zero and second category, while the other was simultaneously first category and nonmeasurable. Although Erdös did not use this terminology, one of his groups is a Lusin set and the other a Sierpinski set. Some small modifications of his theorems yield our Lemmas 5 and 6.

Lemma 5. *Under the assumption of the continuum hypothesis, there exists a Lusin set $G \subseteq R$ which has the following properties:*

(1) *G is a topological group, and*

(2) *G contains sets N_1 and N_2 such that $N_1 \cap N_2 = \varnothing$, N_1 and N_2 are categorically dense in R, and $N_1 \cup N_2$ is linearly independent over Q, the set of rationals.*

Lemma 6. *Under the assumption of the continuum hypothesis, there exists a Sierpinski set $G \subseteq R$ which has the following properties:*

(1) *G is a topological group, and*

(2) *G contains sets N_1 and N_2 such that $N_1 \cap N_2 = \varnothing$, N_1 and N_2 have full outer measure in every measurable set, and $N_1 \cup N_2$ is linearly independent over Q.*

The following is a well-known theorem of Sierpinski and Zygmund, the proof of which may be found in [7].

Lemma 7. *There exists a function $g : R \to R$ such that $g|M$ is not continuous for every set M which has cardinality c.*

Brown and Prikry, in [1], constructed two functions, one of which is universally measurable and has the property that its restriction to every second category set is not continuous, while the other is B_r measurable and has the property that its restriction to every set with positive outer measure is not continuous. In some sense our examples are not as strong as theirs; however, ours have the additional property of being symmetric on large sets.

Example 8. Under the assumption of the continuum hypothesis, there exists a universally measurable function $f : R \to R$ which is symmetric on a second category set, yet $f|M$ is not continuous for every second category set M which has the Baire property in the wide sense. Note that for such f, $S(f)\backslash C(f)$ is second category.

Proof. Let G, N_1, and N_2 be as in Lemma 5, and let $g : R \to R$ be as in Lemma 7.

Now we want to construct f which is symmetric on N_2. Let $A_0 = N_1$. For each positive integer n, let $A_n = (2N_2 - A_{n-1})\backslash(\bigcup_{i=0}^{n-1} A_i)$. Let $x \in A_n$ for some n. Then x has a unique representation in terms of $N_1 \cup N_2$ since $N_1 \cup N_2$ is linearly independent over Q. Let $p(x)$ be the only element of N_1 that appears in this representation of x. Now we define f in the following manner: $f(x) = (-1)^n g(p(x))$ if $x \in A_n$ for some n, otherwise let $f(x) = 0$. In particular, note that f is zero on N_2.

Now we want to show that f is symmetric on N_2. Let $h > 0$ and $x \in N_2$. If both $x + h$ and $x - h$ belong to the complement of $(\bigcup_{i=0}^{\infty} A_i)$, then we are done. Therefore, let us assume that one of $x + h$ or $x - h$ belongs to some A_n. Without loss of generality, let us assume that $x + h \in A_n$. Then $h \in -x + A_n \Rightarrow -h \in x - A_n \Rightarrow x - h \in 2x - A_n$. Note that $x - h \in 2x - A_n$ implies that $x - h \in A_{n-1}$ or $x - h \in A_{n+1}$ and that $p(x-h) = p(x+h)$. So we have that

$$f(x-h) = (-1)^n g(p(x-h)) = (-1)^n g(p(x+h)) = (-1) f(x+h).$$

Therefore, $|f(x+h) + f(x-h) - 2f(x)| = 0$ for every $x \in N_2$ and $h \in R$.

That f is universally measurable follows from the fact that f is zero on the complement of the U_0 set G. (G is U_0 because G is a Lusin set.)

Let M be a second category set which has the Baire property in the wide sense. Let I be a closed interval such that M contains a G_δ set which is dense in I. We want to show that $f|M$ is not continuous. To obtain a contradiction, assume that $f|M$ is continuous. Since N_1 is categorically dense in R and M contains a dense G_δ set subset of I, $M \cap N_1$ is second category. The cardinality of $M \cap N_1$ is c because we are assuming the continuum hypothesis. This implies that $f|(M \cap N_1) = g|(M \cap N_1)$ is continuous, contradicting that g is a Sierpinski-Zygmund function, and completes the proof.

Example 9. Under the assumption of the continuum hypothesis, there exists a B_r function $f: R \to R$ which is symmetric on a set with positive outer measure, yet $f|M$ is not continuous for every positive measure set M. Note that for such f, $S(f)\backslash C(f)$ has positive outer measure.

Proof. Let G, N_1, and N_2 be as in Lemma 6, and let $g: R \to R$ be as in Lemma 7.

We now define f exactly as in the previous example. It follows in the same manner that for every $x \in N_2$ and $h \in R$ we have

$$|f(x+h) + f(x-h) - 2f(x)| = 0.$$

That f has the Baire property in the restricted sense follows from the fact that f is zero on the complement of the AFC set G. (G is AFC because G is a Sierpinski set.)

Let M be a set which has positive measure. We want to show that $f|M$ is not continuous. To obtain a contradiction, assume that $f|M$ is continuous. Since N_1 has the property that it has full outer measure in every set with positive outer measure, $M \cap N_1$ has positive outer measure. The cardinality of $M \cap N_1$ is c because we are assuming the continuum hypothesis. This implies that $f|(M \cap N_1) = g|(M \cap N_1)$ is continuous, contradicting that g is a Sierpinski-Zygmund function, and completes the proof.

Lemma 10. *If C is a Cantor set which is linearly independent over Q, then the group G generated by C over Q, i.e., the set of all finite linear combinations of C with rational coefficients, is of the first category and measure zero.*

Proof. Let C be a Cantor set which is linearly independent over Q, and let G be the group generated by C over Q.

First we will give a proof assuming that C is not a maximal linearly independent set over Q. Consider $M \equiv r_1 C + r_2 C + \cdots + r_n C$ where $r_1, r_2, \ldots, r_n$ are rational numbers. M is a compact set, and since C is not maximal, M has to be of the first category, for otherwise $M - M$ would contain an interval and the rational multiples of $M - M$ would cover the line, contradicting that C is not maximal. M also has measure zero, for otherwise $M - M$ would contain an interval and the rational multiples of $M - M$ would cover the line, contradicting that C is not maximal. G is the countable union of such M's. Therefore G is first category and measure zero.

Now let us consider the general case. C can be written as the union of a singleton set $\{p\}$ and a countable collection of Cantor sets $\{F_k\}$, where $F_{k+1} \supseteq F_k$ for each k and none of the F_k's contain p. Let G_k be the group generated by F_k over Q and let $H = \bigcup G_k$. Each of the G_k is of the first category and measure zero because F_k is not maximal. Since H is the countable union of first category measure zero sets, H is of the first category and measure zero. Note that $G = \bigcup_{r\in Q} H + rp$. Therefore, G is of the first category and measure zero.

Theorem 11. *There are* 2^c *many functions* $f: R \to R$ *such that*

(1) f *is Lebesgue measurable,*
(2) f *has the Baire property in the wide sense,*
(3) $S(f)\backslash C(f)$ *has cardinality* c.

Proof. Let C be a Cantor set such that C is linearly independent over Q. The existence of such a set follows from a result of von Neumann [6].

Let D be a countable dense subset of C. For each $K \subseteq C\backslash D$, we define $f_K : R \to R$ in the following fashion: Let $A_0 = D$. For each positive integer n, let $A_n = (2K - A_{n-1})\backslash(\bigcup_{i=0}^{n-1} A_i)$. Let $x \in A_n$ for some n. Each $x \in A_n$ has a unique representation in terms of $K \cup D$ because $K \cup D$ is linearly independent over Q. Now let $f_K(x) = 1$ if $x \in A_n$ for some even n, $f_K(x) = -1$ if $x \in A_n$ for some odd n, and $f_K(x) = 0$ otherwise. Note that f is zero on K and G^c, where G is the group generated by finite linear combinations of C with rational coefficients.

Now we want to show that f is symmetric on K. Let $h > 0$ and $x \in K$. If both $x + h$ and $x - h$ belong to the complement of $(\bigcup_{i=0}^{\infty} A_i)$, then we are done. Therefore, let us assume that one of $x + h$ or $x - h$ belongs to some A_n. Without loss of generality, let us assume that $x + h \in A_n$. Then $h \in -x + A_n \Rightarrow -h \in x - A_n \Rightarrow x - h \in 2x - A_n$. Since each $x \in A_n$ has a unique representation in terms of $K \cup D$ and $x - h \in 2x - A_n$, we have that $x - h \in A_{n-1}$ or $x - h \in A_{n+1}$. Therefore, $|f_K(x+h) + f_K(x-h) - 2f_K(x)| = 0$ for every $x \in K$ and $h \in R$.

G is of the first category and measure zero by Lemma 10. Each f_K is Lebesgue measurable and has the Baire property in the wide sense because f_K is zero on the complement of G.

The last thing that we need to show is that if $K, L \subseteq C\backslash D$ and $K \neq L$, then $f_K \neq f_L$. To see this, consider $x \in K\backslash L$ or $x \in L\backslash K$ and $d \in D$. Then one of $f_K(2x - d)$ and $f_L(2x - d)$ is zero while the other is 1 or -1.

Since there are 2^c many sets $K \subseteq C\backslash D$ which have cardinality c, the collection of all functions that satisfy properties (1)–(3) of our theorem has cardinality 2^c.

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