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BOUNDARY BEHAVIOR OF GENERALIZED POISSON INTEGRALS FOR THE HALF-SPACE AND THE DIRICHLET PROBLEM FOR THE SCHRÖDINGER OPERATOR

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ABSTRACT. The boundary properties are investigated for the generalized Poisson integral

$$u(X) = \int_{\mathbb{R}^n} k(X, y) f(y)\, dy,$$

where X is a point of the upper half-space $\mathbb{R}^{n+1}_+$, $f \in L^{\mathbf{p}}(\mathbb{R}^n)$, $1 \le \mathbf{p} \le \infty$, and the kernel k has some special properties. Our results imply the known boundary properties of the harmonic Poisson integrals on the half-space. As an application we derive a solution of the Dirichlet problem for the operator $-\Delta + c(X)$, $X \in \mathbb{R}^{n+1}_+$, with boundary data $f \in L^{\mathbf{p}}(\mathbb{R}^n)$.

1. INTRODUCTION AND STATEMENT OF RESULTS

In their well-known paper [1] Fefferman and Stein extended the classical theory of Hardy spaces $\mathbf{H}^{\mathbf{p}}$ on harmonic functions in the half-space

$$\mathbb{R}^{n+1}_+ = \{X = (x, x_{n+1}) : x \in \mathbb{R}^n,\ x_{n+1} > 0\}, \qquad n \ge 1.$$

In this question the Poisson integral plays an essential role. The Laplace operator Δ has constant coefficients; hence the Poisson integral, i.e., a normal derivative of the Green function for the half-space, depends on the difference of arguments and consequently in the harmonic case the Poisson integral is a convolution. In the same article [1] the authors considered the boundary behavior of more general convolution integrals.

In the next step the problem appears to investigate the boundary behavior of nonconvolution integrals of the kind

$$(1) \qquad u(X) = \int_{\mathbb{R}^n} k(X, y) f(y)\, dy, \qquad X \in \mathbb{R}^{n+1}_+,$$

which in this context it is natural to call the generalized Poisson integrals.

If k in (1) is the Poisson kernel for the Laplacian, then the following result is known for the function (1) (see [2, Chapters 3 and 7]).

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Theorem A. *Let* $f \in L^{\mathbf{p}}(\mathbb{R}^n)$, $1 \leq \mathbf{p} \leq \infty$, *and let* $B(x, r)$ *be a ball in* $\mathbb{R}^n$ *with center* x *and radius* r, ω_n *the volume of* $B(0, 1)$, *and* $(Mf)(x) = \sup_{r>0}(\omega_n r^n)^{-1} \int_{B(x,r)} |f(y)|\, dy$ *the maximal function. Then*

1°. $\sup_{0<x_{n+1}<\infty} |u(X)| \leq (Mf)(x)\ \forall x \in \mathbb{R}^n$, *as before, here* $X = (x, x_{n+1})$;
2°. $\lim_{x_{n+1}\to 0} u(X) = f(x)$ *for Lebesgue almost every* $x \in \mathbb{R}^n$;
3°. *for* $\mathbf{p} < \infty$, $\lim_{x_{n+1}\to 0} \|u(X) - f(x)\| = 0$.

Here and in the sequel $\|\cdot\|$ denotes the norm in $L^{\mathbf{p}}(\mathbb{R}^n)$.

Actually in [2] the generalization of Theorem A, particularly when the Poisson kernel is replaced by any approximate identity, was proved. In addition the statement was proved in [2] that in 1° and 2° the condition $x_{n+1} \to 0$ (i.e., $X \to x$ along the normal to the boundary $\mathbb{R}^n$ of $\mathbb{R}^{n+1}_+$) may be interchanged to the tending $X \to x$ as X belongs to the cone

$$\Gamma_\alpha(x_0) = \{X \in \mathbb{R}^{n+1}_+ : |x - x_0| < \alpha x_{n+1}\}, \qquad \alpha > 0,\ x_0 \in \mathbb{R}^n.$$

Our aim is to extend Theorem A on the integrals (1) where the kernel $k(X, y)$ is defined and measurable on the Cartesian product $\mathbb{R}^{n+1}_+ \times \mathbb{R}^n$. We investigate the boundary behavior of (1) and apply our results to the Dirichlet problem for the Schrödinger operator $-\Delta + c(X)I$ in the half-space, I being the identity operator. Now we state our results. All proofs will be given in §2.

Proposition 1. *Fix a point* $x \in \mathbb{R}^n$. *Suppose the kernel* $k(X, y)$ *has the summable majorant* $\psi(x_{n+1}, \cdot) \in L(\mathbb{R}^n)$, *depending only on* $|x-y|$: $|k(X, y)| \leq \psi(x_{n+1}, |x - y|)$ *for all* $X = (x, x_{n+1})$, $0 < x_{n+1} < h$, *with some* $h > 0$. *Let* $\psi(x_{n+1}, r)$ *decrease monotonically for* $0 < r < \infty$ *and*

$$A(x_{n+1}) \equiv \int_{\mathbb{R}^n} \psi(x_{n+1}, |y|)\, dy = O(1) \quad \text{as } x_{n+1} \to 0.$$

If $f \in L^{\mathbf{p}}(\mathbb{R}^n)$, $1 \leq \mathbf{p} \leq \infty$, *then the following is valid for the function* (1):

$$\limsup_{x_{n+1}\to 0} |u(X)| \leq A(Mf)(x),$$

where $A = \limsup_{x_{n+1}\to 0} A(x_{n+1})$.

Corollary. *If the conditions of the proposition are satisfied with the same* ψ *at almost every point* $x \in \mathbb{R}^n$ *and* $1 < \mathbf{p} \leq \infty$, *then*

$$\limsup_{x_{n+1}\to 0} \|u(\cdot, x_{n+1})\| \leq A\|f\|, \qquad A = \text{const}.$$

If $\sup_{x_{n+1}>0} A(x_{n+1}) < \infty$ *and* $\mathbf{p} > 1$, *then*

$$\sup_{x_{n+1}>0} \|u(\cdot, x_{n+1})\| \leq A_{\mathbf{p}}\|f\|.$$

Proposition 2. *Let the conditions of the previous proposition be satisfied for Lebesgue almost every* $x \in \mathbb{R}^n$. *Besides let there exist the limit*

$$\lim_{x_{n+1}\to 0} \int_{\mathbb{R}^n} k(X, y)\, dy = 1$$

for Lebesgue a.e. $x \in \mathbb{R}^n$ *and the limit*

$$\lim_{x_{n+1}\to 0} \int_{|x-y|\geq\delta} |k(X, y)|\, dy = 0 \tag{2}$$

for every $\delta > 0$. *Then there exists the limit*

$$\lim_{x_{n+1}\to 0} u(X) = f(x) \quad \text{a.e. on } \mathbb{R}^n .$$

Remark. If in Propositions 1 and 2 the condition $x_{n+1} \to 0$ is replaced by $\Gamma_\alpha(x_0) \ni X \to x_0$, then all the assertions will remain valid as $\Gamma_\alpha(x_0) \ni X \to x_0$.

Proposition 3. *Suppose all conditions of Proposition* 2 *are satisfied and, in addition, the condition* (2) *is strengthened to*

$$\lim_{x_{n+1}\to 0} \int_{|y|\geq\delta} \psi(x_{n+1}, |y|)\, dy = 0 \quad \forall\delta > 0 .$$

If $1 \leq \mathbf{p} < \infty$ *and* $|\int_{\mathbb{R}^n} k(X, y)\, dy - 1| \leq \text{const} < \infty$ *uniformly in* $X \in \mathbb{R}^n \times (0, h)$, *then there exists the limit*

$$\lim_{x_{n+1}\to 0} \|u(X) - f(x)\| = 0 .$$

Evidently all our conjectures are fulfilled for the classical Poisson kernel

$$\gamma_{n+1} x_{n+1}\{|x-y|^2 + x_{n+1}^2\}^{-(n+1)/2}, \qquad \gamma_{n+1} = \text{const};$$

hence Propositions 1–3 imply Theorem A. Our assumptions are valid too for the kernels

$$(x_{n+1}^2/(|x-y|^2 + x_{n+1}^2))^{\lambda n/2} \cdot x_{n+1}^{-n}$$

and

$$(x_{n+1}/(|x-y| + x_{n+1}))^{\lambda n} \cdot x_{n+1}^{-n}$$

with $\lambda > 1$, and Proposition 1 implies the case $z = 0$ and real $\lambda > 1$ of Lemma 3.3 by Johnson [3].

Now let us consider the operator $L_c = -\Delta + c(X)I$, where the function (potential) $c(X) \geq 0$ in $\mathbb{R}_+^{n+1}$ and such that $c \in L^s$ in some neighbourhood of each finite point $X \in \mathbb{R}_+^{n+1} \cup \mathbb{R}^n$ with certain $s > (n+1)/2$ for $n \geq 3$ and $s = 2$ for $n = 1, 2$. Besides we suppose that $c(X)$ has summable majorant depending only on $|X - y|$ in some vicinity of any boundary point $y \in \mathbb{R}^n$. Under these assumptions it is known that the operator L_c on $L_2(\mathbb{R}_+^{n+1})$ has a Green function $G(X, Y)$ in $\mathbb{R}_+^{n+1}$ with analytic properties necessary in the sequel. Consequently Propositions 1–3 and the results of [4, 5] imply the following.

Theorem. *Suppose* $f \in L^{\mathbf{p}}(\mathbb{R}^n)$, $1 \leq \mathbf{p} \leq \infty$. *Then the Dirichlet problem*

$$\text{(3)} \qquad \begin{cases} -\Delta u + c(X)u, & X \in \mathbb{R}_+^{n+1}, \\ u(x) = f(x) & \text{a.e. on } \mathbb{R}^n \end{cases}$$

has the solution

$$u(X) = \int_{\mathbb{R}^n} \frac{\partial G(X, y)}{\partial n(y)} f(y)\, dy, \qquad X \in \mathbb{R}_+^{n+1},$$

which satisfies all the conclusions of Propositions 1–3. *Here* $\partial/\partial n$ *is a derivative along an inner normal to* $\mathbb{R}^n$. *Moreover, this solution* u *is Hölder continuous with exponent* $2 - n/s$ *if* $n/2 < s \leq n$ *and its gradient* ∇u *is Hölder continuous with exponent* $1 - n/s$ *if* $s > n$ *in all points* $X \in \mathbb{R}_+^{n+1}$. *For potentials under consideration this assertion makes more precise Simon's results about Hölder* *continuity of the solutions to the Schrödinger equation* $L_c u = 0$ [6, *Theorems* C.2.4 *and* C.2.5, *p.* 497].

Of course this solution of problem (3) is not unique. The uniqueness is valid only with a priori growth estimates of the solution at infinity; this question will be considered elsewhere.

2. Demonstrations

We essentially use some ideas of Stein's book [2].

Proof of Proposition 1. By assumption

$$I \equiv \int_{\mathbb{R}^n} \psi(x_{n+1}, |y-x|)\, dy < \infty .$$

If we introduce spherical coordinates in $\mathbb{R}^n$ with pole x, then $I = n\omega_n \int_0^\infty r^{n-1}\psi(x_{n+1}, r)\, dr$. The monotonicity of ψ implies $r^n\psi(x_{n+1}, r) \to 0$ as $r \to 0$ and $r \to \infty$. Now integration by parts leads to the equality $I = \omega_n \int_0^\omega r^n d\{-\psi(x_{n+1}, r)\}$. Denote (see [2])

$$\lambda(r) = \int_{|t-x|=r} f(t)\, d\sigma(t)$$

and

$$\Lambda(r) = \int_{|x-y|\le r} |f(y)|\, dy = \int_0^r t^{n-1}\lambda(t)\, dt .$$

Then $\Lambda(r) \le \omega_n r^n (Mf)(x)$. Hence if $(Mf)(x) < \infty$, then for $0 < x_{n+1} < h$,

(4) $$\lim \Lambda(r)\psi(x_{n+1}, r) = 0, \qquad r \to 0,\ r \to \infty .$$

Now we can estimate the function u:

$$|u(X)| \le \int_{\mathbb{R}^n} |k(X, y)||f(y)|\, dy \le \int_0^\infty \Lambda(r)\, d_r\{-\psi(x_{n+1}, r)\}$$

with regard to (4). From this we have

$$|u(X)| \le \omega_n (Mf)(x) \int_0^\infty r^n\, d_r\{-\psi(x_{n+1}, r)\},$$

and after back integration by parts the inequality $|u(X)| \le A(x_{n+1})(Mf)(x)$ follows. To finish the proof it remains to let $x_{n+1} \to 0$. Q.E.D.

Proof of Proposition 2. Transform the difference

$$\begin{aligned} u(X) - f(x) = &\int_{|y-x|<\delta} k(X, y)\{f(y) - f(x)\}\, dy \\ &+ \int_{|y-x|\ge\delta} k(X, y)\{f(y) - f(x)\}\, dy + f(x)\left\{\int_{\mathbb{R}^n} k(X, y)\, dy - 1\right\} \\ \equiv &\ I_1 + I_2 + I_3 \end{aligned}$$

and estimate every term separately.

Let x be a point of the Lebesgue set of the function f; then $|f(x)| < \infty$. If x does not belong to other exceptional sets of zero measure (see the statement of the proposition), i.e., x belongs to the set of full measure in $\mathbb{R}^n$, then $\lim_{x_{n+1}\to 0} I_3 = 0$ by virtue of (2). Similarly, $\lim_{x_{n+1}\to 0} f(x) \int_{|y-x|\ge\delta} k(X, y)\, dy$

$= 0$. We estimate the integral $I_{21} = \int_{|y-x|\geq\delta} k(X, y)f(y)\,dy$ by the Hölder inequality:

$$|I_{21}| \leq \|f\| \left\{ \int_{|y-x|\geq\delta} |k(X, y)|^{\mathbf{q}}\, dy \right\}^{1/\mathbf{q}}, \qquad \mathbf{q} = \mathbf{p}/(\mathbf{p}-1) \geq 1.$$

The inequality $|k(X, y)| < 1$ is valid asymptotically by virtue of the monotonicity and integrability of the majorant ψ. Hence $|k(X, y)|^{\mathbf{q}} \leq |k(X, y)|$ asymptotically and

$$|I_{21}| \leq \text{const}\|f\| \left\{ \int_{|y-x|\geq\delta} |k(X, y)|\, dy \right\}^{1/\mathbf{q}} \to 0 \quad \text{as } x_{n+1} \to 0.$$

To estimate I_1 we introduce the function as in [2]

$$g(y) = \begin{cases} f(y) - f(x) & \text{if } |y-x| < \delta, \\ 0 & \text{if } |y-x| \geq \delta. \end{cases}$$

For every $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon)$ such that $(Mg)(x) < \varepsilon$ because x is the point of the Lebesgue set of f. Therefore Proposition 1 implies $|I_1| \leq A(Mg)(x) < A\varepsilon$, where ε may be arbitrarily small. Q.E.D.

Proof of Proposition 3. We have $\|u(X) - f(x)\| \leq \|I_1\| + \|I_2\| + \|I_3\|$. First

$$\begin{aligned} |I_1| &= \left| \int_{|y|<\delta} k(X, y)\{f(y+x) - f(x)\}\, dy \right| \\ &\leq \int_{|y|\leq\delta} \psi(x_{n+1}, |y|)|f(y+x) - f(x)|\, dy, \end{aligned}$$

but $\psi(x_{n+1}, r)$ does not depend on x; hence

$$\|I_1\| \leq \int_{|y|<\delta} \psi(x_{n+1}, r)\|f(y+\cdot) - f(\cdot)\|\, dy.$$

It is known that the relation $\Delta(y) \equiv \|f(y+\cdot) - f(\cdot)\| = o(1)$ as $|y| < \delta$, $\delta \to 0$. Hence for arbitrary fixed $\varepsilon > 0$ one can choose $\delta > 0$ such that $\Delta(y) < \varepsilon$ and consequently

$$\|I_1\| \leq \varepsilon \int_{|y|<\delta} \psi(x_{n+1}, |y|)\, dy \leq A(x_{n+1})\varepsilon \leq A\varepsilon.$$

Now we fix $\delta > 0$ and estimate

$$\begin{aligned} \|I_2\| &\leq \left\| \int_{|y-x|\geq\delta} \psi(x_{n+1}, |y-x|)|f(y) - f(x)|\, dy \right\| \\ &\leq \int_{|y|\geq\delta} \psi(x_{n+1}, |y|)\|f(y+\cdot) - f(\cdot)\|\, dy \\ &\leq 2\|f\| \int_{|y|\geq\delta} \psi(x_{n+1}, |y|)\, dy = o(1), \qquad x_{n+1} \to 0. \end{aligned}$$

To estimate $\|I_3\|$ (now $\mathbf{p} < \infty$) we use the representation (see [2]) $f = f_1 + f_2$, where f_1 is a continuous function with compact support and $\|f_2\| < \varepsilon$. Hence $\sup|f_1| < \infty$ and the function with compact support

$$\left\{ |f_1(x)| \left| \int_{\mathbb{R}^n} k(X, y)\, dy - 1 \right| \right\}^{\mathbf{p}}$$

has a summable majorant. Now from the Lebesgue theorem about dominated convergence we have

$$\lim_{x_{n+1}\to 0}\int_{\mathbb{R}^n}|f_1(x)|^{\mathbf{p}}\left|\int_{\mathbb{R}^n}k(X, y)\,dy-1\right|^{\mathbf{p}}dx=0.$$

Finally,

$$\left\|f_2(x)\left(\int_{\mathbb{R}^n}k(X, y)\,dy-1\right)\right\|\leq \mathrm{const}\|f_2\|\leq \mathrm{const}\,\varepsilon. \quad \text{Q.E.D.}$$

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