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NONREGULAR EXTREME POINTS IN THE SET OF MINKOWSKI ADDITIVE SELECTIONS

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ABSTRACT. A function $s: \mathscr{K}^n \to \mathbb{R}^n$, defined on the family $\mathscr{K}^n$ of all compact convex and nonempty sets in $\mathbb{R}^n$, is called a Minkowski additive selection, provided $s(A+B) = s(A) + s(B)$ and $s(A) \in A$, whenever $A, B \in \mathscr{K}^n$. We confirm the conjecture [6] that there exist extremal selections which are not regular (s is regular if $s(A) \in \operatorname{ext} A$, $A \in \mathscr{K}^n$).

Let $\mathscr{K}^n$ denote the family of all convex compact nonempty subsets of $\mathbb{R}^n$. A mapping $T: \mathscr{K}^n \to \mathbb{R}^n$ is called *Minkowski additive*, or simply *additive*, if $T(A+B) = T(A) + T(B)$ for all $A, B \in \mathscr{K}^n$. Let $\mathscr{L}^n$ be the vector space of all additive mappings equipped with the weakest topology under which all evaluations $\mathscr{L}^n \ni T \to T(A)$, $A \in \mathscr{K}^n$, are continuous. It can be easily seen that the set $\mathscr{S}^n \subset \mathscr{L}^n$ of all selections, i.e., the mappings having the property $T(A) \in A$, is convex and compact. Let $\mathscr{E}^n$ be the set of all extreme points of $\mathscr{S}^n$. An element $s \in \mathscr{S}^n$ is called *regular* if $s(A) \in \operatorname{ext} A$, where $\operatorname{ext} A$ denotes the set of all extremal points of A. The set of all regular selections will be denoted by $\mathscr{R}^n$. Obviously, if $s \in \mathscr{R}^n$, then $s \in \mathscr{E}^n$.

Živaljević [6] conjectured that there exist nonregular extremal points in $\mathscr{S}^n$. The following result confirms this supposition.

Theorem. *For every $n \geq 2$, there exists a closed face of $\mathscr{S}^n$ disjoint with $\mathscr{R}^n$.*

To prove the theorem, we shall need some additional notions and definitions. Let us define the *support function* $h(A, x)$ of $A \in \mathscr{K}^n$ at x as follows:

$$h(A, x) = \sup\{\langle a, x\rangle : a \in A\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product. By Ω_k we denote the *Stiefel manifold* of k-frames; that is, the ordered k-tuples $\omega = (x_1, \dots, x_k)$ of orthonormal vectors in $\mathbb{R}^n$. For $\omega \in \Omega_k$, we define the ω-face $V_\omega(A)$ of A inductively: Let $V_{x_1}(A) = \{a \in A : \langle a, x_1\rangle = h(A, x_1)\}$ and suppose that we have already defined $V_{\omega'}$ for $\omega' = (x_1, \dots, x_{k-1})$. Then $V_\omega(A) = V_{x_k}(V_{\omega'}(A))$. Subsequently, for $V_\omega(A)$, we define its position vector $H_\omega(A)$ as follows:

$$H_{x_1}(A) = h(A, x_1)x_1, \qquad H_\omega(A) = H_{\omega'}(A) + h(V_{\omega'}(A), x_k)x_k.$$

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It was the main result of [6] that $\mathscr{R}^n = \{H_\omega : \omega \in \Omega_n\}$. Basic properties of the face mappings can be found in [2, 5].

Suppose now that $A \in \mathscr{K}^2$. Let us denote by $s_0(A)$ the center of the smallest rectangle containing A, which has its sides parallel to the coordinate axes. It is easy to observe that $s_0(A) \in A$. Moreover, $s_0(A)$ can be expressed by the formula

$$\begin{aligned} s_0(A) = &\tfrac{1}{2}(h(A, e_1) - h(A, -e_1))e_1 \\ &+ \tfrac{1}{2}(h(A, e_2) - h(A, -e_2))e_2 \end{aligned}$$

where e_1, e_2 denote vectors of the standard basis in $\mathbb{R}^2$. Obviously, s_0 is an additive selection on $\mathscr{K}^2$. This selection has already been mentioned in [1, 4].

Proposition. *The minimal closed face of* $\mathscr{S}^2$ *containing* s_0 *is disjoint with* $\mathscr{R}^2$.

Proof. For every pair $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, we define the triangle $T(\varepsilon_1, \varepsilon_2) = \operatorname{conv}\{0, \varepsilon_1 e_1, \varepsilon_2 e_2\}$. It is clear that $s_0(T(\varepsilon_1, \varepsilon_2)) = (\varepsilon_1 e_1 + \varepsilon_2 e_2)/2$. Hence for every selection s belonging to the minimal closed face containing s_0 we have

$$s(T(\varepsilon_1, \varepsilon_2)) \in [\varepsilon_1 e_1, \varepsilon_2 e_2]. \tag{$*$}$$

On the other hand, it can be easily seen that no regular point H_ω, $\omega \in \Omega_2$, can satisfy all the relations $(*)$ resulting when ε_1 and ε_2 run over $\{-1, 1\}$. □

Proof of the theorem. Let us regard $\mathbb{R}^2$ as embedded into $\mathbb{R}^n$ in an obvious manner. Let $\omega \in \Omega_{n-2}$ consist of elements orthogonal to $\mathbb{R}^2$. It is clear that $V_\omega(A) - H_\omega(A) \in \mathbb{R}^2$ for every $A \in \mathscr{K}^n$. Consequently, the following mapping is well defined:

$$s_\omega(A) = s_0(V_\omega(A) - H_\omega(A)) + H_\omega(A).$$

It is an easy exercise to prove that s_ω is an additive selection on $\mathscr{K}^n$. Furthermore, it follows from the proposition that the minimal face containing s_ω is disjoint with $\mathscr{R}^n$. □

Question. Is it true that $s_\omega \in \mathscr{E}^n$?

For further information on the topic of extremal selections the reader is referred to the more extensive study [3].

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