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BANACH ALGEBRAS WHICH ARE NOT WEDDERBURNIAN

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ABSTRACT. Let A be a Banach algebra with radical R. In 1951 Feldman exhibited an example in which it is impossible to find a closed subalgebra K of A such that $A = K \oplus R$. We provide other examples. Feldman's algebra is commutative, but these examples are, in general, not commutative.

1. INTRODUCTION

In [5, p. 85] Glaeser called a Banach algebra A *Wedderburnian* if A is the direct sum of its radical R and a closed subalgebra K of A. In [2] Bade and Curtis called such a Banach algebra strongly decomposable. If A is finite dimensional then a classical result of Wedderburn shows that A is Wedderburnian. In [4] Feldman provided an example where A is not Wedderburnian. This algebra was studied in detail in [1]. For another example see [5]. These examples are commutative. Our aim is to provide noncommutative examples which occur rather naturally. Some instances arise as follows. Let G be an infinite compact topological group with identity e. Let $C(G)$ be the set of all complex-valued continuous functions on G taken as an algebra with convolution as its multiplication. Let $\|f\|_2$ be the L^2-norm of $f \in C(G)$. The norm

$$|||f||| = \max(\|f\|_2, |f(e)|)$$

is a normed algebra norm on $C(G)$. The completion A of $C(G)$ in this norm is not Wedderburnian. The Feldman example can be identified with the completion of the socle of $C(G)$, for G the reals modulo one, in the norm $|||f|||$.

Other examples arising from algebras of operators on Hilbert space are given. In particular, the completion of the trace class [9, p. 37] of Schatten in an appropriate norm is not Wedderburnian.

2. PRELIMINARY THEORY

We adopt the following notation. Let B be a Banach algebra in the norm $\|x\|$ and E be a Banach space in the norm $\|\xi\|_E$. Let T be a linear mapping of B into E satisfying

$$\|T(xy)\|_E \le \|x\| \, \|y\| \tag{1}$$

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for all x, y in B. We let A be the set of all elements of the form $x+\xi$, where $x \in B$ and $\xi \in E$, made into an algebra under the rules $(x+\xi)+(y+\eta) = (x+y)+(\xi+\eta)$, $a(x+\xi) = (ax+a\xi)$, and $(x+\xi)(y+\eta) = xy$ for all $x, y \in B$, $\xi, \eta \in E$, and scalars a.

We define a norm on A by

$$|||x+\xi||| = \max(\|x\|, \|\xi - T(x)\|_E).$$

In view of (1) we see that $|||x+\xi|||$ is a normed algebra norm on A.

2.1. **Lemma.** *A is a Banach algebra in the norm $|||x+\xi|||$.*

Proof. Let $\{x_n + \xi_n\}$ be a Cauchy sequence in A. Then $\{x_n\}$ is a Cauchy sequence in B and $\{T(x_n) - \xi_n\}$ is a Cauchy sequence in E. Hence there exists $y \in B$ and $\eta \in E$ where $\|x_n - y\| \to 0$ and $\|T(x_n) - \xi_n - \eta\|_E \to 0$. One readily checks that, in A, the sequence $x_n + \xi_n$ has $y + [T(y) - \eta]$ as its limit.

We denote the radical of A by R.

2.2. **Lemma.** *If B is semisimple then $R = E$.*

Proof. Clearly $E \subset R$. Note that B is a two-sided ideal in A. Therefore, $R \cap B$ is the radical of B so that $R \cap B = (0)$. Let $x + \xi \in R$ where $x \in B$ and $\xi \in E$. Then $x \in R \cap B$ so that $x = 0$.

2.3. **Lemma.** *Suppose that T is discontinuous on a linear subspace W of B. If E is finite dimensional then the closure of W in A must contain a nonzero element of E.*

Proof. There exists a sequence $\{x_n\}$ in W where $\|x_n\| \to 0$ and $\|T(x_n)\|_E = 1$ for each $n = 1, 2, \ldots$. As E is finite dimensional, there is a subsequence $\{y_n\}$ of $\{x_n\}$ and some $\xi \neq 0$ in E such that $\|\xi - T(y_n)\|_E \to 0$. Then, since

$$|||y_n + \xi||| = \max(\|y_n\|, \|\xi - T(y_n)\|_E),$$

we see that $-\xi$ is in the closure of W in A.

2.4. **Theorem.** *Suppose that B is semisimple and that E is finite dimensional. Suppose that W is a two-sided ideal in B and that T is discontinuous on W^2. Then the completion V of W in the norm*

$$|||x||| = \max(\|x\|, \|T(x)\|_E)$$

is a Banach algebra which is not Wedderburnian.

Proof. By Lemma 2.1, V is just the closure of W in the Banach algebra A. Also V is a two-sided ideal in A so that, by Lemma 2.2, the radical S of V is $V \cap E$.

Suppose that $V = K \oplus S$ where K is a subalgebra of V. Since $sv = vs = 0$ for all $s \in S$ and $v \in V$, we have $K \supset V^2 \supset W^2$. Hence, by Lemma 2.3, the closure of K in V must contain a nonzero element of S. Consequently V is not Wedderburnian.

3. Examples from harmonic analysis

Let G be an infinite compact topological group with identity e and normalized Haar measure $m(E)$. We consider $C(G)$ in the sup norm and $L^2(G)$ in the L^2-norm $\|f\|_2$ as Banach algebras with convolution $f * g$ as the multiplication. If f and g are in $L^2(G)$ then $f * g$ lies in $C(G)$ by [6, p. 295]. Therefore, the socle $\mathfrak{S}$ of $L^2(G)$ lies in $C(G)$, and $\mathfrak{S}$ is also the socle of $C(G)$. As $C(G)$ is a dual algebra [7, Theorem 15], $\mathfrak{S}$ is dense in $C(G)$ as well as $L^2(G)$.

We use the standard description of $\mathfrak{S}$ provided by the Peter-Weyl theorem. Let Λ be the set of equivalence classes of finite-dimensional irreducible representations of G. For each $\alpha \in \Lambda$ we select an irreducible unitary representation R^α in the class α. Suppose $R^\alpha(t)$ is the n_α by n_α matrix $(D^\alpha_{ij}(t))$. Then $\mathfrak{S}$ consists of all linear combinations of the functions D^α_{ij}, $\alpha \in \Lambda$, $i, j = 1, \dots, n_\alpha$. The functions $n_\alpha^{1/2} D^\alpha_{ij}$ form an orthonormal basis for $L^2(G)$. Also $D^\alpha_{ij} * D^\beta_{rs} = 0$ if $\alpha \neq \beta$ and

$$n_\alpha D^\alpha_{ij} * n_\alpha D^\alpha_{rs} = n_\alpha \delta_{jr} D^\alpha_{is}, \tag{2}$$

where δ_{jr} is the Kronecker delta.

Let $\mathfrak{D}$ be the set of all linear combinations of the "diagonal" entries D^α_{ii}, $\alpha \in \Lambda$, $i = 1, \dots, n_\alpha$. The convolution of two different diagonal entries is zero and each D^α_{ii} is a scalar multiple of an idempotent. Therefore, $\mathfrak{D}$ is a commutative subalgebra of $C(G)$.

3.1. **Lemma.** *Consider $\mathfrak{D}$ as a normed algebra in the L^2-norm $\|f\|_2$. Then the linear functional $f \to f(e)$ is discontinuous on $\mathfrak{D}$.*

Proof. Let

$$f = \sum_{k=1}^{r} k^{-1} D^{\alpha_k}_{i_k i_k}$$

be in $\mathfrak{D}$ where the $D^{\alpha_k}_{i_k i_k}$ are different diagonal entries. Then $f(e) = \sum_{k=1}^r k^{-1}$ and $\|f\|_2 = \sum_{k=1}^r k^{-2} n_{\alpha_k}^{-1/2}$.

3.2. **Lemma.** *The closure of $\mathfrak{D}$ in either $C(G)$ or $L^2(G)$ is semisimple.*

Proof. Note that $v = n_\alpha^{1/2} D^\alpha_{ii}$ is an idempotent generator of a minimal one-sided ideal of $C(G)$ or $L^2(G)$. Let W be the closure of $\mathfrak{D}$ and z be in the radical of W. We have, as W is commutative, that $vz = zv = vzv$ is a scalar multiple of the idempotent v and is in the radical of W. Hence $D^\alpha_{ii} * z = 0$. It follows from (2) that $D^\alpha_{rs} * z = 0$ for all $\alpha \in \Lambda$ and $r, s = 1, \dots, n_\alpha$. Hence $\mathfrak{S} * z = z * \mathfrak{S} = (0)$. Since $\mathfrak{S}$ is dense in $C(G)$ and $L^2(G)$ and these are semisimple, we see that $z = 0$.

3.3. *Notation.* The functional $f \to f(e)$ which is defined naturally on $C(G)$ can be extended to a linear functional $\phi(f)$ defined on $L^2(G)$ by [10, Theorem 1.71-A, p. 40].

3.4. **Lemma.** *For any $f, g \in L^2(G)$ we have*

$$|\phi(f * g)| \leq \|f\|_2 \|g\|_2.$$

Proof. As noted earlier, $f * g \in C(G)$. Therefore,

$$|\phi(f * g)| = |f * g(e)| \leq \|f\|_2 \|g\|_2$$

by Schwarz's inequality.

3.5. **Theorem.** *Let K be either the closure of $\mathfrak{D}$ in $L^2(G)$ or any two-sided ideal of $L^2(G)$ containing $\mathfrak{D}$. Then the completion of K in the norm*

$$|||f||| = \max[\|f\|_2, |\phi(f)|]$$

is a Banach algebra which is not Wedderburnian.

Proof. Clearly $\mathfrak{D}^2 = \mathfrak{D}$. Theorem 3.5 follows from Lemmas 3.1, 3.2, and 3.4 together with Theorem 2.4. The special case $K = C(G)$ was mentioned in §1.

For the case $K = \mathfrak{S}$ we have a specific result.

3.6. **Corollary.** *Let f be a typical element of $\mathfrak{S}$ where*

$$f = \sum_{k=1}^{r} a_k D_{i_k j_k}^{\alpha_k}.$$

Here each a_k is a scalar and no two D_{ij}^{α} agree in all of α, i, and j. Then the completion of $\mathfrak{S}$ in the norm

$$|||f||| = \max\left\{\left(\sum_{k=1}^{r} |a_k|^2/n_{\alpha_k}\right)^{1/2}, \left|\sum_{k=1}^{r} a_k \delta_{i_k j_k}\right|\right\}$$

is not Wedderburnian.

Proof. We use Theorem 3.5 together with $D_{ij}^{\alpha}(e) = \delta_{ij}$.

If G is abelian each $m_\alpha = 1$ and each $\delta_{ij} = 1$. Here $\mathfrak{S}$ is the set of linear combinations of the continuous characters of G.

3.7. **Corollary.** *Let G be an abelian compact group whose character group $\widehat{G}$ is denumerably infinite: $\widehat{G} = \{\gamma_1, \gamma_2, \dots\}$. Then the completion of $\mathfrak{S}$ in the norm*

$$\left|\left|\left|\sum_{k=1}^{r} a_k \gamma_k\right|\right|\right| = \max\left\{\left(\sum_{k=1}^{r} |a_k|^2\right)^{1/2}, \left|\sum_{k=1}^{r} a_k\right|\right\}$$

is not Wedderburnian.

For G the reals modulo one we have, except for a difference in notation, the Feldman example [4].

4. Examples from operator theory

Let $B(H)$ be the algebra of all bounded linear operators on a separable infinite-dimensional Hilbert space H. Let $\{\phi_n\}$ be an orthonormal basis for H. As in Schatten's book [9] (see also [3, Chapter 1]) we consider the Schmidt class B_2 and the trace-class B_1 of operators on H. B_2 is the set of all $T \in B(H)$ for which $\sum_j \|T(\phi_j)\|^2 < \infty$. This sum is finite and the same if $\{\phi_n\}$ is replaced by another orthononormal basis $\{\psi_n\}$. As shown in [9], B_2 is a Banach $*$-algebra in the norm

$$\|T\|_2 = \left[\sum_j \|T(\phi_j)\|^2\right]^{1/2}.$$

Also $\|T\|_2 = \|T^*\|_2$ for all $T \in B_2$.

Let $|T|$ be the unique positive square root of T^*T. The trace-class B_1 is the set of all $T \in B(H)$ for which $\sum_j(|T|(\phi_j), \phi_j) < \infty$. Again this sum is finite and the same if $\{\phi_n\}$ is replaced by another orthonormal basis $\{\psi_n\}$. As shown in [9], B_1 is a Banach $*$-algebra under the norm

$$\|T\|_1 = \sum_j (|T|(\phi_j), \phi_j).$$

Furthermore, B_1 is the set of all products TU for $T, U \in B_2$, and the elements of B_1 all have a finite trace

$$\operatorname{tr}(U) = \sum_j (U(\phi_j), \phi_j), \qquad U \in B_1,$$

again independent of the choice of the orthonormal basis. By [10, 1.71-A, p. 40], $\operatorname{tr}(U)$ can be extended to be a linear functional $TR(U)$ on all of B_2.

We note that (see [9]) the common socle of B_1 and B_2 is the set $F(H)$ of all $U \in B(H)$ with finite-dimensional range.

4.1. **Lemma.** *For $U, V \in B_2$ we have*

$$|TR(UV)| \le \|U\|_2\|V\|_2.$$

Proof. As noted above, $UV \in B_1$. Therefore,

$$\begin{aligned}|TR(UV)| &= \left|\sum_j (UV(\phi_j), \phi_j)\right| \le \sum_j |(V(\phi_j), U^*(\phi_j))| \\ &\le \sum_j \|V(\phi_j)\|\,\|U^*(\phi_j)\| \le \|V\|_2\|U^*\|_2 = \|V\|_2\|U\|_2.\end{aligned}$$

4.2. **Lemma.** $\operatorname{tr}(U)$ *is discontinuous on $F(H)$ if $F(H)$ is taken in the norm $\|U\|_2$.*

Proof. For each positive integer n we define $W_n \in F(H)$ as follows. Let $W_n(\phi_j) = \phi_j/j$ for $j = 1, \dots, n$ and $W_n(\phi_j) = 0$ for $j > n$. Then

$$\operatorname{tr}(W_n) = \sum_{j=1}^{n} j^{-1} \quad \text{and} \quad \|W_n\|_2 = \sum_{j=1}^{n} j^{-2}.$$

4.3. **Theorem.** *Let K be any two-sided ideal of B_2 which contains $F(H)$. The completion of K in the norm*

$$|||V||| = \max(\|V\|_2, |TR(V)|)$$

is not Wedderburnian.

Proof. Note that $F(H) = [F(H)]^2$. We can then use Lemmas 4.1 and 4.2 to apply Theorem 2.4. The particular case $K = B_1$ was noted in §1.

Consider the specific case $H = l_2$. Any $V \in B(l_2)$ can be described in matrix terms. There corresponds to V an infinite matrix $[v_{rs}]$ so that, for $x = \{x_n\}$ and $y = \{y_n\}$ in l_2, $V(x) = y$ if and only if

$$y_r = \sum_{s=1}^{\infty} v_{rs}x_s, \qquad r = 1, 2, \dots.$$

For V we have

$$\|V\|_2 = \left[\sum_j \sum_j |v_{ij}|^2\right]^{1/2}, \qquad \operatorname{tr}(V) = \sum_j v_{jj}.$$

In these terms B_2 is the set of all $V \in B(l_2)$ for which $\sum_j \sum_i |v_{ij}|^2 < \infty$, and although there seems to be no simple description of B_1 in matrix terms (see [8, p. 107]), $F(l_2)$ is easily described as all $V \in B_2$ for which the column vectors of the matrix $[v_{rs}]$ lie in a finite-dimensional subspace of l_2.

4.4. **Corollary.** *The completion of $F(l_2)$ in the normed algebra norm*

$$|||V||| = \max\left\{\left[\sum_j \sum_i |v_{ij}|^2\right]^{1/2}, \left|\sum_j v_{jj}\right|\right\}$$

is not Wedderburnian.

Added in proof

Interesting examples of commutative Banach algebras not Wedderburnian were given by G. F. Bachelis and S. Saeki, *Banach algebras with uncomplemented radical*, Proc. Amer. Math. Soc. **100** (1987), 271–273.

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