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ON SETS OF VITALI'S TYPE

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ABSTRACT. We consider the classical Vitali's construction of nonmeasurable subsets of the real line $\mathbb{R}$ and investigate its analogs for various uncountable subgroups of $\mathbb{R}$. Among other results we show that if G is an uncountable proper analytic subgroup of $\mathbb{R}$ then there are Lebesgue measurable and Lebesgue nonmeasurable selectors for $\mathbb{R}/G$.

0. INTRODUCTION

In this paper we investigate some properties of selectors related to various subgroups of the additive group of reals $\mathbb{R}$. The first example of a selector of this type was constructed by Vitali in 1905 (using an uncountable form of the Axiom of Choice) by partitioning $\mathbb{R}$ into equivalence classes with respect to the subgroup $\mathbb{Q}$ of all rationals. Vitali's set shows the existence of Lebesgue nonmeasurable sets and the existence of sets without the Baire property. Moreover, the construction of the Vitali set stimulated the formulation of further problems and questions in the classical measure theory, for instance, the question about invariant extensions of Lebesgue measure, the general problem concerning the existence of universal measures on $\mathbb{R}$, the question about effective existence of Lebesgue nonmeasurable subsets of $\mathbb{R}$, etc.

If G is an arbitrary subgroup of $\mathbb{R}$ then we call a set $X \subseteq \mathbb{R}$ a G-selector if X is a selector of the family of all equivalence classes $\mathbb{R}/G$. In this paper we shall consider properties of G-selectors for various subgroups of $\mathbb{R}$.

First we consider G-selectors in the case of groups $G \subseteq \mathbb{R}$ with a good descriptive structure. Then we discuss G-selectors for arbitrary groups $G \subseteq \mathbb{R}$. The third section of the paper is devoted to certain natural groups $G \subseteq \mathbb{R}$ in some models of set theory.

In this paper we use standard set theoretical notation. As usual, ω denotes the set of all natural numbers and at the same time the first infinite cardinal number. ω_1 denotes the first uncountable cardinal and $\mathfrak{c}$ denotes the cardinality of the continuum. If X is a set then $|X|$ denotes the cardinality of X. If f is a function then $\mathrm{dom}(f)$ is the domain of f. By $\mathbb{L}$ we denote the class of all Lebesgue measurable subsets of $\mathbb{R}$, so we have $\mathbb{L} = \mathrm{dom}(\mu_1)$, where μ_1 is one-dimensional Lebesgue measure on the real line. $\mathbb{L}_0$ denotes the σ-ideal of all Lebesgue measure zero subsets of $\mathbb{R}$.

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In the paper we restrict ourselves to consider only the questions related to the measurability of G-selectors. Analogously, one can consider the question of Baire property of those sets; in fact, the arguments below work also in the case of Baire property and the corresponding results can be proved. We do not intend to discuss these problems.

1. The case of groups $G \subseteq \mathbb{R}$ with good descriptive structure

We shall start with the following question: If a given group $G \subseteq \mathbb{R}$ is good in the descriptive sense (for instance, G is Borel, or more generally, projective), then how good may a G-selector be? In order to answer this question we recall two well-known lemmas.

Lemma 1 (Steinhaus). *If A and B are μ_1-measurable subsets of $\mathbb{R}$ such that $\mu_1(A) > 0$ and $\mu_1(B) > 0$ then the set $A + B = \{a + b : a \in A \,\&\, b \in B\}$ has a nonempty interior.*

For the proof see [Ox].

Lemma 2 (Mycielski). *If Z is a Lebesgue measurable subset of the plane $\mathbb{R}^2$ such that $(\mu_1 \times \mu_1)(\mathbb{R}^2 \setminus Z) = 0$ then there exists a perfect set $X \subseteq \mathbb{R}$ such that $X \times X \subseteq Z \cup \{(x, x) : x \in \mathbb{R}\}$.*

The proof of this result can be found in [My2]. Let us recall at this place that an analogous result for category holds, too (see [My1]).

Lemma 3 (Mycielski). *If Z is a comeager subset of the plane $\mathbb{R}^2$ then there exists a perfect set $X \subseteq \mathbb{R}$ such that $X \times X \subseteq Z \cup \{(x, x) : x \in \mathbb{R}\}$.*

Let G be a subgroup of $\mathbb{R}$. It follows immediately from Lemma 1 that if $G \in \mathbb{L}$ then either $G \in \mathbb{L}_0$ or $G = \mathbb{R}$.

Moreover, it follows from Lemma 2 that if $G \in \mathbb{L}_0$ then $|\mathbb{R}/G| = \mathfrak{c}$.

After these preliminary remarks let us consider G-selectors for a group G, which belongs to the classical projective hierarchy of subsets of $\mathbb{R}$.

Theorem 1. *Let a nondiscrete proper subgroup $G \subseteq \mathbb{R}$ belong to the projective class Σ^1_n. If $\Sigma^1_n \subseteq \mathbb{L}$ then no G-selector belongs to the class Σ^1_n.*

Proof. Suppose that there exists a G-selector Y in the class Σ^1_n. Let us define a canonical surjection $f : \mathbb{R} \to Y$ by the formula

$$f(x) = y \quad \text{iff} \quad (y \in Y \,\&\, x - y \in G).$$

Using the fact that the class Σ^1_n is closed under finite intersections, finite products, and continuous images, it is easy to see that f is a Σ^1_n-measurable function, i.e., that the preimage of any open set by the function f is in Σ^1_n. From our assumption $\Sigma^1_n \subseteq \mathbb{L}$ it follows that f is a Lebesgue measurable function. At the same time f is a G-invariant function, i.e., $f(x + g) = f(x)$ for all $x \in \mathbb{R}$ and $g \in G$.

Remembering that G is a dense subgroup of $\mathbb{R}$ and using the well-known fact that the Lebesgue measure μ_1 is metrically transitive, we can find $y_0 \in \mathbb{R}$ such that for almost all $x \in \mathbb{R}$ we have $f(x) = y_0 = \text{const}$.

But, on the other hand, the assumption $G \in \Sigma^1_n \subseteq \mathbb{L}$ gives us $G \in \mathbb{L}_0$ and

$$f^{-1}(y_0) = G + y_0 \in \mathbb{L}_0.$$

This contradiction finishes the proof of the theorem. □

Remark 1. Since the inclusion $\Sigma_1^1 \subseteq \mathbb{L}$ holds in ZFC, we can directly apply Theorem 1 to the class of all analytic subsets of $\mathbb{R}$.

Remark 2. If the conjunction of Martin's Axiom and the negation of the Continuum Hypothesis holds then, as we know, the inclusion $\Sigma_2^1 \subseteq \mathbb{L}$ is true and so we can apply Theorem 1 in this case, too.

Remark 3. The assumption $\Sigma_n^1 \subseteq \mathbb{L}$ is essential in the formulation of Theorem 1. Indeed, if Gödel's Axiom of Constructibility holds, then a well-known result states that there exist $\mathbb{Q}$-selectors in the class Σ_2^1 (where $\mathbb{Q}$ is the group of all rational numbers).

Now let us discuss the problem of Lebesgue measurability of G-selectors for good subgroups of $\mathbb{R}$. The well-known construction from [EKM] gives a decomposition of $\mathbb{R}$ into a direct sum of two subgroups G_1 and G_2 such that G_1 is an uncountable analytic set and G_2 is a Lebesgue measure zero set of cardinality continuum. The following theorem throws new light on this result.

Theorem 2. *If G is an uncountable analytic subgroup of $\mathbb{R}$ then there exists a measurable G-selector.*

Proof. Let G be an uncountable analytic subgroup of $\mathbb{R}$ and let

$$S = \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R}: x - y \in G\}.$$

Notice that S is a continuous preimage of an analytic set; hence S is an analytic subset of the plane $\mathbb{R}^2$ with uncountable vertical sections. By Mokobodzki's theorem (see [Mo]), there exists a set $X \in \mathbb{L}_0$ such that $\mathbb{R}\setminus S^{-1}[X] \in \mathbb{L}_0$, but $S^{-1}[X] = G + X$. Choose $X' \subseteq X$ such that $G + X' = G + X$ and $|X' \cap Z| \leq 1$ for any set $Z \in \mathbb{R}/G$. Next extend the set X' to a set $Y \supseteq X'$ such that $Y \in \mathbb{L}_0$ and Y is a G-selector. □

The proof of Theorem 2 can be based also on a result from [EKM], which says that for every nonempty perfect subset A of $\mathbb{R}$ there exists a Lebesgue measure zero subset B of $\mathbb{R}$ such that $A + B = \mathbb{R}$. Moreover, we can conclude that if a given group $G \subseteq \mathbb{R}$ contains a nonempty perfect subset then there exists a Lebesgue measurable G-selector.

The situation for measurability of G-selectors is more interesting if we replace the Lebesgue measure μ_1 by some other measures extending μ_1. To illustrate this we first prove the following lemma.

Lemma 4. *Suppose that G is an uncountable analytic subgroup of $\mathbb{R}$ and $A \subseteq \mathbb{R}$ is a Lebesgue measurable set such that $|A \cap (G + t)| \leq \omega$ for every $t \in \mathbb{R}$. Then A is a measure zero set.*

Proof. Assume that A and G are as above. Let G' be any uncountable F_σ-subgroup of G. Then we also have $|A \cap (G' + t)| \leq \omega$ for each $t \in \mathbb{R}$. Hence we may assume from the beginning that the given group G is a Borel subgroup of $\mathbb{R}$. It is also clear that we may assume that A is a Borel set, too. Let H be any countable subset of G dense in $\mathbb{R}$ and let $A' = A + H$. Then $\mathbb{R}\setminus A'$ has Lebesgue measure zero and $|A' \cap (G + t)| \leq \omega$ for each $t \in \mathbb{R}$. Hence we may assume from the beginning that $\mathbb{R}\setminus A$ has measure zero.

Suppose that Martin's Axiom and the negation of the Continuum Hypothesis hold. Let T be a subset of G with cardinality ω_1. Then the intersection of the family $\{A-t: t \in T\}$ has full Lebesgue measure and, in particular, is nonempty. Let $x \in \bigcap\{A-t: t \in T\}$. Then $T+x \subseteq A$, hence $|A \cap (G+x)| \geq \omega_1$, so we obtain a contradiction.

Suppose now that the conjunction of Martin's Axiom and the negation of the Continuum Hypothesis does not hold. Notice that the sentence " $\mathbb{R}\setminus A \in \mathbb{L}_0 \& G$ is an uncountable subgroup of $\mathbb{R}$ " is absolute. Let V' be a generic extension of the universe V in which Martin's Axiom and the negation of the Continuum Hypothesis hold. Then in V' there exists $t \in \mathbb{R}$ such that $|A \cap (G+t)| \geq \omega_1$, but the sentence " $(\exists t \in \mathbb{R})(|A \cap (G+t)| \geq \omega_1)$ " is absolute, too. Hence the same holds in the universe V. Thus also in this case we obtain a contradiction. □

Theorem 3. *If G is an uncountable analytic subgroup of $\mathbb{R}$ then there exists a σ-aditive measure ν on $\mathbb{R}$ such that:*

(a) *ν is invariant under all isometric transformations of the real line $\mathbb{R}$;*
(b) *ν extends the Lebesgue measure μ_1;*
(c) *any G-selector is ν-measurable.*

Proof. Let $\mathbb{I}$ be the σ-ideal generated by all G-selectors. Obviously this ideal is invariant under all isometric transformations of $\mathbb{R}$. It follows from the last lemma that the interior Lebesgue measure of any set from the ideal $\mathbb{I}$ is zero. Hence, using a standard technique of constructing extensions of invariant measures (see, for example, [K]), we can extend Lebesgue measure μ_1 on the σ-field generated by Lebesgue measurable sets and the ideal $\mathbb{I}$. This gives us the required measure ν. □

2. The case of arbitrary groups $G \subseteq \mathbb{R}$

Let Card denote the class of all cardinal numbers. We shall consider the following partial order on the class Card $\times$ Card :

$$(\kappa, \lambda) \leq (\kappa', \lambda') \quad \text{iff} \quad (\lambda > \lambda') \vee (\lambda = \lambda' \;\&\; (\kappa \leq \kappa')).$$

We apply this order to measure the size of subgroups of the real line $\mathbb{R}$. For the sake of simplicity we denote a pair (κ, λ) of cardinal numbers by κ/λ. If G is a subgroup of $\mathbb{R}$ then let $\|G\| = |G|/|\mathbb{R}/G|$.[1] To clarify this notation let us observe that

$$\|\{0\}\| = \frac{1}{\mathfrak{c}} < \|\mathbb{Q}\| = \frac{\omega}{\mathfrak{c}} < \frac{\mathfrak{c}}{\mathfrak{c}} < \|\mathbb{R}\| = \frac{\mathfrak{c}}{1}.$$

For any family $\mathbb{S}$ of subsets of $\mathbb{R}$, we define the following cardinal numbers:

$$\begin{aligned} \operatorname{cov}(\mathbb{S}) &= \min\{|\mathbb{T}|: \mathbb{T} \subseteq \mathbb{S} \;\&\; \textstyle\bigcup \mathbb{T} = \mathbb{R}\}, \\ \operatorname{non}(\mathbb{S}) &= \min\{|X|: X \subseteq \mathbb{R} \;\&\; X \notin \mathbb{S}\}, \\ \operatorname{cof}(\mathbb{S}) &= \min\{|\mathbb{T}|: \mathbb{T} \subseteq \mathbb{S} \;\&\; (\forall A \in \mathbb{S})(\exists B \in \mathbb{T})(A \subseteq B)\}. \end{aligned}$$

In fact, Vitali proved that if the group G is nondiscrete and $|G| < \operatorname{cov}(\mathbb{L}_0)$ then any G-selector is Lebesgue nonmeasurable.

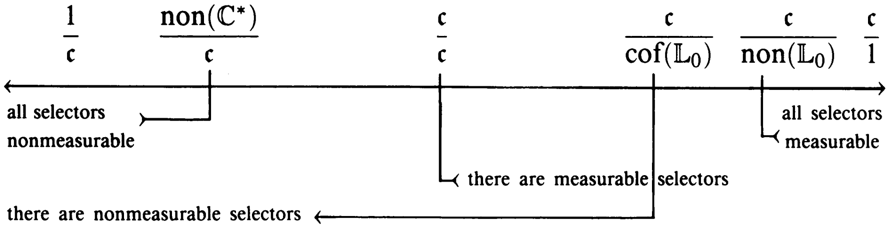
[1] We are aware that the definition introduced above may appear too sophisticated and artificial. We know that many particular instances of its usage can be formulated in a simpler way; however, the application of this particular definition allows one to understand comprehensibly the results contained in the paper.

Let $\mathbb{C}^*$ denote the family of all subsets $A \subseteq \mathbb{R}$ such that for every $B \in \mathbb{L}_0$ we have $A + B \neq \mathbb{R}$.[2] It is easy to check that $\mathrm{non}(\mathbb{C}^*) \geq \mathrm{cov}(\mathbb{L}_0) \geq \omega_1$. Let $\mathbb{K}$ denote the σ-ideal of all sets of first Baire category in $\mathbb{R}$.

Theorem 4. *Let G be a nondiscrete proper subgroup of $\mathbb{R}$. Then*

(1) *if $\|G\| < [\mathrm{non}(\mathbb{C}^*)]/\mathfrak{c}$ then every G-selector is Lebesgue nonmeasurable;*
(2) *if $\|G\| > \mathfrak{c}/\mathrm{non}(\mathbb{L}_0)$ then every G-selector is Lebesgue measurable;*
(3) *if $\|G\| \leq \mathfrak{c}/\mathrm{cof}(\mathbb{L}_0)$ then there exists a Lebesgue nonmeasurable G-selector;*
(4) *if $\|G\| > \mathfrak{c}/\mathfrak{c}$ then there exists a Lebesgue measurable G-selector.*

We can summarize[3] *these results as follows:*



Proof. Parts (1) and (2) follow directly from definitions. Let us begin with the proof of part (3).

Suppose that $|\mathbb{R}/G| \geq \mathrm{cof}(\mathbb{L}_0)$. Let $\{X_\alpha : \alpha < \mathrm{cof}(\mathbb{L}_0)\}$ be an enumeration of a basis of the ideal $\mathbb{L}_0$. By induction we define a transfinite sequence $\{x_\alpha : \alpha < \mathrm{cof}(\mathbb{L}_0)\}$ of reals such that for every $\alpha < \beta < \mathrm{cof}(\mathbb{L}_0)$ we have $x_\alpha \notin X_\alpha$ and $x_\alpha - x_\beta \notin G$. Note that any such sequence gives us a nonmeasurable set, which can be extended to a nonmeasurable G-selector.

Suppose that $\zeta < \mathrm{cof}(\mathbb{L}_0)$ and that $\{x_\alpha : \alpha < \zeta\}$ is defined. If

$$X_\zeta \cup (\{x_\alpha : \alpha < \zeta\} + G) \neq \mathbb{R}$$

then we can continue our construction. Otherwise we have

$$\mu_1(\mathbb{R} \setminus (\{x_\alpha : \alpha < \zeta\} + G)) = 0.$$

Note now that for each infinite group H and a subset $A \subseteq H$ such that $|A| < |H|$ there exists $h \in H$ such that $A + h \subseteq H \setminus A$. Applying this fact to the group $\mathbb{R}/G$ we can find an element $h \in \mathbb{R}$ such that the set $\{x_\alpha : \alpha < \zeta\} + G + h$ is disjoint with the set $G + \{x_\alpha : \alpha < \zeta\}$, but this is impossible, since Lebesgue measure μ_1 is invariant under translations of $\mathbb{R}$. Consequently we see that our induction will not stop before $\mathrm{cof}(\mathbb{L}_0)$. Thus (3) is proved.

To prove (4) assume that $|\mathbb{R}/G| < \mathfrak{c}$. It follows from Mycielski's Lemma 3 above that $G \notin \mathbb{K}$. Let $\{A, B\}$ be any fixed partition of $\mathbb{R}$ such that $A \in \mathbb{L}_0$ and $B \in \mathbb{K}$. Obviously for each $t \in \mathbb{R}$ we have $(t + G) \cap A \neq \varnothing$. This fact implies the existence of a G-selector $X \subseteq A$. Finally, it is clear that $X \in \mathbb{L}_0$. $\square$

[2]The notation $\mathbb{C}^*$ is used by set theorists working with reals. Any association with special classes of algebras is meaningless at this place.

[3] At the referee's suggestion we add here a more classical version of the assumptions of Theorem 4. (1) $\|G\| < [\mathrm{non}(\mathbb{C}^*)]/\mathfrak{c}$ iff $|G| < \mathrm{non}(\mathbb{C}^*)$; (2) $\|G\| > \mathfrak{c}/\mathrm{non}(\mathbb{L}_0)$ iff $|\mathbb{R}/G| < \mathrm{non}(\mathbb{L}_0)$; (3) $\|G\| \leq \mathfrak{c}/\mathrm{cof}(\mathbb{L}_0)$ iff $|\mathbb{R}/G| \geq \mathrm{cof}(\mathbb{L}_0)$; (4) $\|G\| > \mathfrak{c}/\mathfrak{c}$ iff $|\mathbb{R}/G| < \mathfrak{c}$.

Theorem 5. *Suppose that* κ, λ *are infinite cardinals such that* $\kappa \cdot \lambda = 2^\omega$ *and* $\kappa/\lambda \leq \mathfrak{c}/\mathrm{non}(\mathbb{L}_0)$. *Then there exists a subgroup* G *of* $\mathbb{R}$ *with a nonmeasurable* G*-selector such that* $\|G\| = \kappa/\lambda$.

Proof. To begin, let us consider the case when $\mathfrak{c} > \lambda \geq \mathrm{non}(\mathbb{L}_0)$. Notice that in this case $\kappa = \mathfrak{c}$. Let $A \subseteq \mathbb{R}$ be a Lebesgue nonmeasurable set of cardinality λ. We apply Zorn's lemma to the family

$$\mathbb{S} = \{G \subseteq \mathbb{R}: (G \text{ is a subgroup of } \mathbb{R}) \;\&\; \mathbb{Q}\cdot G = G \;\&\; (A - A) \cap G = \{0\}\}$$

ordered by inclusion and we get a $\subseteq$-maximal group G from $\mathbb{S}$. It is clear that A can be extended to a Lebesgue nonmeasurable G-selector, so $|\mathbb{R}/G| \geq |A| = \lambda$. It remains to show that $|\mathbb{R}/G| \leq \lambda$ (and, in particular, we shall get $|G| = \mathfrak{c} = \kappa$). We will show that $\mathbb{Q}\cdot((A-A)+G) = \mathbb{R}$, which finishes the proof in this case, because then

$$|\mathbb{R}/G| = |\mathbb{Q}\cdot((A-A)+G)/G| = |(\mathbb{Q}\cdot(A-A)+G)/G| = |\mathbb{Q}\cdot(A-A)| \leq \omega\cdot\lambda.$$

Hence, suppose that $t \in \mathbb{R}\setminus\mathbb{Q}\cdot((A-A)+G)$. Let $H = \{g + q\cdot t: g \in G \;\&\; q \in \mathbb{Q}\}$. Then H is a subgroup of $\mathbb{R}$, $\mathbb{Q}\cdot H = H$, and G is a proper subgroup of H. Hence there are $a, b \in A$ and $h \in H\setminus\{0\}$ such that $a - b = h$. So $a - b = g + q\cdot t$ for some $g \in G$ and $q \in \mathbb{Q}$. Note that $q \neq 0$, so $t = ((a-b)-g)/q$ and we obtain a contradiction.

It remains to consider the case when $\lambda = \mathfrak{c}$. In this case $\omega \leq \kappa \leq \mathfrak{c}$. Let G be any subgroup of $\mathbb{R}$ such that $\|G\| = \kappa/\lambda$. Notice that for each Borel set $X \subseteq \mathbb{R}$ with $\mu_1(X) > 0$ we have

$$|\{H \in \mathbb{R}/G: H \cap X \neq \varnothing\}| = \mathfrak{c}.$$

This fact easily follows from Steinhaus's theorem mentioned above. Using this fact one can construct by induction two transfinite sequences $\{x_\alpha: \alpha < \mathfrak{c}\}$ and $\{y_\alpha: \alpha < \mathfrak{c}\}$ such that

$$x_\alpha - x_\beta \notin G, \quad y_\alpha - y_\beta \notin G \qquad (\alpha < \beta < \mathfrak{c}),$$
$$\{x_\alpha: \alpha < \mathfrak{c}\} \cap \{y_\alpha: \alpha < \mathfrak{c}\} = \varnothing,$$

and both sets $\{x_\alpha: \alpha < \mathfrak{c}\}$ and $\{y_\alpha: \alpha < \mathfrak{c}\}$ have a full outer Lebesgue measure. Consequently, each of these sets can be extended to a nonmeasurable G-selector. This finishes the proof of Theorem 5. □

Remark 4. Suppose that Martin's Axiom holds. Then we have $\mathrm{non}(\mathbb{C}^*) = \mathrm{non}(\mathbb{L}_0) = \mathrm{cof}(\mathbb{L}_0) = \mathfrak{c}$. If $\|G\| < \mathfrak{c}/\mathfrak{c}$ then, by Theorem 4(1), every G-selector is nonmeasurable. If $\|G\| < \mathfrak{c}/\mathfrak{c}$ then, by Theorem 4(2), every G-selector is measurable. If $\|G\| = \mathfrak{c}/\mathfrak{c}$ then, by Theorem 4(3), there exists a nonmeasurable G-selector G. Moreover, if G is an uncountable analytic group then there exists a measurable G-selector, too (see Theorem 2). It is worthwhile to notice that for some groups G with $\|G\| = \mathfrak{c}/\mathfrak{c}$ there are no measurable G-selectors. In fact, we have the following

Theorem 6. *Suppose that Martin's Axiom holds. Then there exists a group* G *with no measurable* G*-selectors such that* $\|G\| = \mathfrak{c}/\mathfrak{c}$.

Proof. Let $\{X_\alpha: \alpha < c\} \subseteq \mathbb{L}_0$ be any basis for the σ-ideal $\mathbb{L}_0$. We shall define two sequences $\{g_\alpha: \alpha < \mathfrak{c}\}$ and $\{t_\alpha: \alpha < \mathfrak{c}\}$ of real numbers such that for each

$\beta < \mathfrak{c}$ we have

(a) $G_\beta + t_\alpha \subseteq \mathbb{R} \backslash X_\alpha$ for every $\alpha < \beta$,
(b) $g_\beta \in \mathbb{R} \backslash G_\beta$,

where G_β is the vector subspace of $\mathbb{R}$, treated as a vector space over $\mathbb{Q}$, generated by $\{g_\alpha : \alpha < \beta\}$.

Note that if $\beta < \mathfrak{c}$ and the sequences $\{g_\alpha : \alpha < \beta\}$ and $\{t_\alpha : \alpha < \beta\}$ are defined, then it is sufficient to take as g_β any element from the set

$$\bigcap\{q \cdot ((\mathbb{R} \backslash X_\alpha) - t_\alpha - h) : \alpha < \beta \ \&\ q \in \mathbb{Q} \backslash \{0\} \ \&\ h \in G_\beta\} \backslash G_\beta$$

and as t_β any element such that $G_{\beta+1} + t_\beta \subseteq \mathbb{R} \backslash X_\beta$.

Now let G be the vector subspace of $\mathbb{R}$ generated by the sequence $\{g_\alpha : \alpha < \mathfrak{c}\}$. The condition (b) implies that $|G| = \mathfrak{c}$. Suppose that $X \in \mathbb{L}_0$ is a G-selector. Then there exists $\alpha < \mathfrak{c}$ such that $X \subseteq X_\alpha$. The condition (a) implies that

$$(G + t_\alpha) \cap X \subseteq (G + t_\alpha) \cap X_\alpha = \varnothing,$$

so X is not a G-selector. This contradiction shows that there are no Lebesgue measurable G-selectors. We can also easily deduce from this that $|\mathbb{R}/G| = \mathfrak{c}$, hence $\|G\| = \mathfrak{c}/\mathfrak{c}$ and, therefore, the theorem is proved. □

Remark 5. Let $\mathscr{M}$ be a model of ZFC + GCH, let κ be a regular cardinal number in $\mathscr{M}$, and let $\mathscr{N}$ be a model obtained from $\mathscr{M}$ by adding κ independent Cohen reals. Carlson proved that in $\mathscr{N}$ the Dual Borel Conjecture holds. Clearly, there exists a G-selector of measure zero provided $G \notin \mathbb{C}^*$. Hence (in $\mathscr{N}$) if $\|G\| > \omega/\mathfrak{c}$ then there exists a measurable G-selector. By Theorem 4, if $\|G\| \leq \mathfrak{c}/\mathfrak{c}$ then there exists a nonmeasurable G-selector and if $\|G\| > \mathfrak{c}/\mathfrak{c}$ then all G-selectors are measurable (to see this, note that $\mathrm{non}(\mathbb{L}_0) = 2^\omega$ in the model $\mathscr{N}$).

Remark 6. The problem of measurability of G-selectors is also completely clear if G is a Borel (or analytic) group. If $\|G\| = \omega/\mathfrak{c}$ and G is nondiscrete then every G-selector is nonmeasurable. If G is infinite and discrete or if $\|G\| = \mathfrak{c}/\mathfrak{c}$ then there are measurable and nonmeasurable G-selectors. The other two cases ($1/\mathfrak{c}$ and $\mathfrak{c}/1$) are trivial.

3. The case of certain groups $G \subseteq \mathbb{R}$ in models of set theory

Now we shall consider the problem of measurability of G-selectors in the case when G is the group of all reals of some submodel of the universe.

The following technical proposition is due to Carlson (see, for instance, [P]).

Lemma 5. *There exists a Borel set $A \subseteq \mathbb{R} \times \mathbb{R}$ such that each of its vertical sections is of Lebesgue measure zero and for every Borel function $f : \mathbb{R} \to \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $\{t \in \mathbb{R} : \langle t, f(t) - x \rangle \notin A\}$ is a first category subset of $\mathbb{R}$.*

Using Lemma 4 we can prove the following proposition.

Theorem 7. *Let $\mathscr{M}$ be a model of ZFC and let c be a Cohen real over model $\mathscr{M}$. Then in $\mathscr{M}[c]$ there exists a measurable $(\mathbb{R} \cap \mathscr{M})$-selector.*

Proof. Let c be a Cohen real over $\mathscr{M}$ and let $b \in \mathscr{M}$ be a Borel code of a subset of $\mathbb{R}^2$, the existence of which is established by Lemma 5. Let B be the set decoded from b in $\mathscr{M}[c]$. Then $B_c \in \mathbb{L}_0$ and for every real $f \in \mathbb{R}$ there exists $x \in \mathbb{R} \cap \mathscr{M}$ such that $f - x \in B_c$. Hence $B_c + (\mathbb{R} \cap \mathscr{M}) = \mathbb{R}$, thus there exists a subset of B_c, which is an $(\mathbb{R} \cap \mathscr{M})$-selector.

The next technical proposition is a slight modification of a result of Krawczyk (see [P]).

Lemma 6. *Suppose that $\mathscr{M}$ is a model of ZFC and r is a Solovay real over $\mathscr{M}$. Let A be a Borel measure zero subset of the real line $\mathbb{R}$ coded in the model $\mathscr{M}[r]$. Then there exists a real number $x \in \mathbb{R}$ such that $((\mathbb{R} \cap \mathscr{M}) + x) \cap A = \varnothing$.*

Using the last lemma we obtain the following result.

Theorem 8. *Let $\mathscr{M}$ be a model of ZFC and let r be a Solovay real over model $\mathscr{M}$. Then in $\mathscr{M}[r]$ every $(\mathbb{R} \cap \mathscr{M})$-selector is Lebesgue nonmeasurable.*

Proof. Suppose that X is a Lebesgue measurable $(\mathbb{R} \cap \mathscr{M})$-selector. Then $X \in \mathbb{L}_0$. Let A be any Borel set from $\mathbb{L}_0$ such that $X \subseteq A$. By Lemma 6 there exists $x \in \mathbb{R}$ such that $((\mathbb{R} \cap \mathscr{M}) + x) \cap A = \varnothing$. So we see that X is not an $(\mathscr{M} \cap \mathbb{R})$-selector. This contradiction proves the theorem. $\square$

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