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STRONGLY EXTREME POINTS AND THE RADON-NIKODÝM PROPERTY

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ABSTRACT. We prove that if K is a bounded and convex subset of a Banach space X and x is a point in K, then x is a strongly extreme point of K if and only if x is a strongly extreme point of $\overline{K}^*$ which is the weak* closure of K in X^{**}. We also prove that a Banach space X has the Radon-Nikodým property if and only if for any equivalent norm on X, the unit ball has a strongly extreme point.

Suppose K is a subset of a Banach space X and $x \in K$. The element x is called an extreme point of K if $x \notin \mathrm{co}(K\backslash\{x\})$, where $\mathrm{co}(K\backslash\{x\})$ is the convex hull of the set $K\backslash\{x\}$. Various kinds of extreme points have been introduced and studied, among them are denting points and strongly extreme points. Denting points can be defined in terms of slices of K which are of the form

$$S(x^*, K, \delta) = \{x \in K : x^*(x) > \sup x^*(K) - \delta\},$$

where δ is a positive number and x^* is an element in X^*, the dual of X. The element x is called a denting point of K if the family of all slices of K containing x is a neighborhood base of x with respect to the relative norm topology on K. It is called a strongly extreme point of K if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any y in X the conditions $d(x+y, K) < \delta$ and $d(x-y, K) < \delta$ imply that $\|y\| < \varepsilon$, where $d(x, K)$ is the distance between x and K. We use $\mathrm{ext}\,K$ (resp. $\text{str-ext}\,K$, $\mathrm{dent}\,K$) to denote the set of the extreme (resp. strongly extreme, denting) points of K. It is obvious that if $x \in \mathrm{dent}\,K$, then $x \in \text{str-ext}\,K$. In addition, it is easy to see that if K is convex and $x \in \text{str-ext}\,K$, then $x \in \mathrm{ext}\,K$. Let $\overline{K}^*$ be the weak* closure of K in X^{**}. An extreme point of K may not be an extreme point of $\overline{K}^*$, even if K is the unit ball of X [5]. On the other hand, we will show that if K is bounded and convex and $x \in K$, then $x \in \text{str-ext}\,K$ if and only if $x \in \text{str-ext}\,\overline{K}^*$ (see Theorem 3).

Two important properties of Banach spaces, namely, the Radon-Nikodým property (RNP) and the Krein-Milman property (KMP), can be defined in terms of denting points and extreme points respectively. The Banach space X is said

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to have the RNP (resp. KMP) if every nonempty bounded closed convex set K in X has a denting (resp. extreme) point [1]. It is unknown whether the RNP and the KMP are equivalent. However, using a result of Huff and Morris [3], it can be proved that X has the RNP if and only if every nonempty bounded closed convex set K in X has an extreme point of $\overline{K}^*$ [1, Corollary 3.76; 4, Remarks, p. 174]. Morris [5] proved that every separable Banach space that contains an isomorphic copy of c_0 admits an equivalent strictly convex norm $|\ |$ such that the unit ball $B_{(X,|\ |)}$ of X has no extreme points of the unit ball $B_{(X^{**},|\ |)}$ of X^{**}. On the other hand, it is known that X has the RNP if and only if for any equivalent norm on X the respective unit ball B_X has a denting point (see, e.g., [1, p. 30]). Thus, as observed by Morris [5], if X has the RNP, then for any equivalent norm on X the respective unit ball B_X has an extreme point of $B_{X^{**}}$. Morris conjectured [5] that the converse is also true. Though we are not able to prove the conjecture in this paper, we will show that X has the RNP, if and only if for any equivalent norm on X the respective unit ball B_X has a strongly extreme point (see Corollary 6).

For our discussion, we will need several equivalent formulations of strongly extreme points listed in Lemma 1. We omit the proof of Lemma 1 because it is straightforward.

Lemma 1. *Suppose K is a subset of a Banach space X and $x \in K$. The following are equivalent:*

(1) $x \in \text{str-ext}\,K$.

(2) *For any sequence $\{x_n\}$ in X, if $\lim_n d(x \pm x_n, K) = 0$, then $\lim_n x_n = 0$.*

(3) *For any sequences $\{x_n\}$ and $\{y_n\}$ in K, if $\lim_n (x_n + y_n)/2 = x$, then $\lim_n x_n = \lim_n y_n = x$.*

(4) *For any $\varepsilon > 0$ there is a $\delta > 0$, such that for any two vectors x_1 and x_2 in K, if $\|(x_1 + x_2)/2 - x\| < \delta$ then $\|x_1 - x_2\| < \varepsilon$.*

Lemma 2 may be used to reduce some problems about general convex sets to problems about symmetric convex sets (see the proof of Theorem 3).

Lemma 2. *Suppose K is a subset of a Banach space X. Let $Sy(X, K)$ be the convex hull of $\{(x, 1), (-x, -1) : x \in K\}$ and let $\overline{Sy}^*(X, K)$ be the weak* closure of $Sy(X, K)$ in the bidual of the direct sum $X \oplus \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.*

(1) *The set $Sy(X, K)$ is symmetric.*

(2) *If K is bounded, then $Sy(X, K)$ is bounded and $\overline{Sy}^*(X, K) = Sy(X^{**}, \overline{\text{co}}^* K)$, where $\overline{\text{co}}^* K$ is the weak* closure of $\text{co}\,K$ in X^{**}.*

(3) *If K is bounded and convex, then $\text{str-ext}\,Sy(X, C) = \{(x, 1), (-x, -1) : x \in \text{str-ext}\,K\}$.*

Proof. (1) and (2) are obvious. Without loss of generality, we assume the norm on $X \oplus \mathbb{R}$ is defined by $\|(x, r)\| = \max\{\|x\|, |r|\}$ for every (x, r) in $X \oplus \mathbb{R}$. Let $A = \{(x, 1) : x \in K\}$, and let $B = -A$. Since $Sy(X, K) = \text{co}(A \cup B)$, we have $\text{str-ext}\,Sy(X, K) \subset A \cup B$. Thus $\text{str-ext}\,Sy(X, K) \subset \text{str-ext}\,A \cup \text{str-ext}\,B$. It is obvious that $\text{str-ext}\,A = \{(x, 1) : x \in \text{str-ext}\,K\}$ and $\text{str-ext}\,B = \{(-x, -1) : x \in \text{str-ext}\,K\}$. Let $M = \sup\{\|z\| : z \in Sy(X, K)\}$, and let $x \in K$. Note that $M \geq 1$. If $(x, 1) \notin \text{str-ext}\,Sy(X, K)$, then there is $\varepsilon > 0$ such that for any

$\varepsilon/2 > \delta > 0$ there are u_1 and u_2 in $Sy(X, K)$ satisfying

$$\|(u_1 + u_2)/2 - (x, 1)\| < \delta/(6M) \quad \text{and} \quad \|u_1 - u_2\| > \varepsilon.$$

For $i = 1$ or 2, there are x_i and y_i in K and t_i in $[0, 1]$ such that $u_i = t_i(x_i, 1) + (1-t_i)(-y_i, -1)$. It follows that $2 - t_1 - t_2 \le \|(u_1+u_2)/2 - (x, 1)\| < \delta/(6M)$. Thus

$$\|u_i - (x_i, 1)\| = (1 - t_i)\|(x_i + y_i, 2)\| < \delta/3.$$

Hence

$$\begin{aligned}\|x_1 - x_2\| = \|(x_1, 1) - (x_2, 1)\| &\ge \|u_1 - u_2\| - \|u_1 - (x_1, 1)\| - \|u_2 - (x_2, 1)\| \\ &> \varepsilon - \delta/3 - \delta/3 > \varepsilon/2\end{aligned}$$

and

$$\begin{aligned}\|(x_1 + x_2)/2 - x\| &= \|[(x_1, 1) + (x_2, 1)]/2 - (x, 1)\| \\ &\le \|(u_1 + u_2)/2 - (x, 1)\| + \|[u_1 - (x_1, 1)]/2\| + \|[u_2 - (x_2, 1)]/2\| \\ &< \delta/(6M) + \delta/6 + \delta/6 < \delta.\end{aligned}$$

Therefore, $x \notin \text{str-ext}\, K$. Similarly if $(-x, -1) \notin \text{str-ext}\, Sy(X, K)$, then $x \notin \text{str-ext}\, K$. Hence $\text{str-ext}\, Sy(X, K) = \{(x, 1), (-x, -1) : x \in \text{str-ext}\, K\}$. Q.E.D.

Theorem 3. *If $K \subset X$ is bounded and convex and $x \in K$, then $x \in \text{str-ext}\, K$ if and only if $x \in \text{str-ext}\, \overline{K}^*$.*

Proof. Since K is a subset of $\overline{K}^*$, if $x \in \text{str-ext}\, \overline{K}^*$ then $x \in \text{str-ext}\, K$. Now suppose $x \in \text{str-ext}\, K$. By Lemma 2, we have $(x, 1) \in \text{str-ext}\, Sy(X, K)$ and $\overline{Sy}^*(X, K) = Sy(X^{**}, \overline{K}^*)$, and $(x, 1) \in \text{str-ext}\, Sy(X^{**}, \overline{K}^*)$ if and only if $x \in \text{str-ext}\, \overline{K}^*$. Passing to $(x, 1)$ and $Sy(X, K)$ if necessary, we may assume that K is also symmetric. Assume that $x \notin \text{str-ext}\, \overline{K}^*$. Then there are $\varepsilon > 0$ and a sequence $\{x_n^{**}\}$ in X^{**} such that $\|x_n^{**}\| > \varepsilon$ and $d(x \pm x_n^{**}, \overline{K}^*) < 1/n$. For each $n \ge 1$, choose $x_n^* \in S_{X^*}$ such that $x_n^{**}(x_n^*) > \varepsilon$, and let $\| \, \|_n$ be the Minkowski functional determined by $K + 1/nB_X$. It is obvious that $B_{(X^{**}, \| \, \|_n)} = \overline{K}^* + 1/nB_{X^{**}}$. Thus $\|x \pm x_n^{**}\|_n < 1$. By the local reflexivity of Banach spaces [2], for each $n \ge 1$ there is a linear operator T_n from $\text{span}\{x, x_n^{**}\}$ to X such that

$$\|T_n(x \pm x_n^{**})\|_n < 1, \quad T_n(x) = x, \quad \text{and} \quad x_n^*(T_n(x_n^{**})) = x_n^{**}(x_n^*).$$

Let $x_n = T_n(x_n^{**})$. Then $\|x_n\| \ge x_n^*(x_n) = x_n^{**}(x_n^*) > \varepsilon$ and $\|x \pm x_n\|_n < 1$. So $x \pm x_n \in K + 1/nB_X$, that is, we have $d(x \pm x_n, K) \le 1/n$. Therefore, $x \notin \text{str-ext}\, K$, which is a contradiction. Hence $x \in \text{str-ext}\, \overline{K}^*$. Q.E.D.

Without assuming K to be bounded, one can prove that if $x \in \text{str-ext}\, K$ then x is an extreme point of $\text{ext}\, \overline{K}^*$ (see [4, Remarks, p. 174] or Lemma 4).

Lemma 4. *Suppose $K \subset X$ is convex and $x \in K$. Consider the following statements*:

(1) *For any sequences $\{x_n\}$ and $\{y_n\}$ in K, if* $\lim_n (x_n + y_n)/2 = x$, *then* $\text{weak-lim}_n\, x_n = \text{weak-lim}_n\, y_n = x$.

(2) *For any nets $\{x_\lambda\}$ and $\{y_\lambda\}$ in K, if* $\text{weak-lim}_\lambda (x_\lambda + y_\lambda)/2 = x$, *then* $\text{weak-lim}_\lambda\, x_\lambda = \text{weak-lim}_\lambda\, y_\lambda = x$.

(3) *The element* x *is an extreme point of* $\overline{K}^*$.
(4) *For any bounded sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* K, *if* $\lim_n(x_n + y_n)/2 = x$, *then* weak-$\lim_n x_n =$ weak-$\lim_n y_n = x$.
(5) *For any bounded nets* $\{x_\lambda\}$ *and* $\{y_\lambda\}$ *in* K, *if* weak-$\lim_\lambda(x_\lambda+y_\lambda)/2 = x$, *then* weak-$\lim_\lambda x_\lambda =$ weak-$\lim_\lambda y_\lambda = x$.

Then (1) *and* (2) *are equivalent and each of them implies* (3). *Statements* (4) *and* (5) *are equivalent and both are implied by* (3). *Thus if, in addition, the set* K *is bounded, then all the above statements are equivalent.*

Proof. It is obvious that (2) implies (1). To prove the converse is true, we assume that there exist some nets $\{x_\lambda\}$ and $\{y_\lambda\}$ in K such that weak-$\lim_\lambda(x_\lambda + y_\lambda)/2 = x$, but $\{x_\lambda\}$ does not converge weakly to x. Passing to subnets if necessary, we may assume that there is x^* in X^* such that $x^*(x_\lambda - x) > 1$. Since weak-$\lim_\lambda(x_\lambda + y_\lambda)/2 = x$, we may assume that $x^*(x - y_\lambda) > 0$. Let $A = \mathrm{co}\{x_\lambda\}$ and $B = \mathrm{co}\{y_\lambda\}$. Then $\inf x^*(A) \geq \max x^*(B)+1$ and there is a sequence z_n in $\mathrm{co}\{(x_\lambda+y_\lambda)/2\}$ such that $\lim_n z_n = x$. Hence there are sequences $\{x_n\}$ in $\mathrm{co}\{x_\lambda\}$ and $\{y_n\}$ in $\mathrm{co}\{y_\lambda\}$ such that $(x_n + y_n)/2 = z_n$. Thus $\lim_n(x_n + y_n)/2 = x$ and $x^*(x_n - y_n) > 1$, which imply that either $\{x_n\}$ or $\{y_n\}$ does not converge weakly to x. Therefore, (1) implies (2).

The proof of the equivalence of (4) and (5) is similar.

Assume that x is not an extreme point of $\overline{K}^*$. Then there are x^{**} and y^{**} in $\overline{K}^*$ such that $x^{**} \neq x \neq y^{**}$ and $x = (x^{**} + y^{**})/2$. Choose x^* in X^* such that $(x^{**} - x)(x^*) > 1$. Then $(x - y^{**})(x^*) > 1$. There exist nets $\{x_\lambda\}$ and $\{y_\lambda\}$ in K such that weak*-$\lim_\lambda x_\lambda = x^{**}$ and weak*-$\lim_\lambda y_\lambda = y^{**}$. Thus weak-$\lim_\lambda(x_\lambda + y_\lambda)/2 = x$, but $\{x_\lambda\}$ does not converge weakly to x. Hence (2) implies (3).

Finally, to show (3) implies (5), we assume that x is an extreme point of $\overline{K}^*$. Suppose $\{x_\lambda\}$, $\{y_\lambda\}$ are two bounded nets in K with weak-$\lim_\lambda(x_\lambda+y_\lambda)/2 = x$. Then $\{x_\lambda\}$ has a weak* cluster point. Let x^{**} be a weak* cluster point of $\{x_\lambda\}$. Then $x^{**} \in \overline{K}^*$ and there is a subnet $\{x_{\lambda(\alpha)}\}$ of $\{x_\lambda\}$ such that weak*-$\lim_\alpha x_{\lambda(\alpha)} = x^{**}$. Since weak-$\lim_\lambda(x_\lambda + y_\lambda)/2 = x$, the weak* limit of $\{y_{\lambda(\alpha)}\}$ exists, say, weak*-$\lim_\alpha y_{\lambda(\alpha)} = y^{**}$. Then $y^{**} \in \overline{K}^*$ and $x = (x^{**}+y^{**})/2$. Since x is an extreme point of $\overline{K}^*$, we can conclude that $x^{**} = x$. Hence weak-$\lim_\lambda x_\lambda =$ weak-$\lim_\lambda y_\lambda = x$. Q.E.D.

Theorem 5. *Suppose* $K_1, K_2 \subset X$ *are closed and convex, and one of them is bounded and* $x \in X$. *Let* K *be the weak* closure of* $K_1 + K_2$ *in* X^{**}. *If* x *is an extreme point of the weak* closure of* K *in* $X^{(4)}$, *the fourth dual of* X, *then* x *is in* $K_1 + K_2$. *In particular, if* x *is a strongly extreme point of the norm closure of* $K_1 + K_2$, *then* x *is in* $K_1 + K_2$.

Proof. It is obvious that the weak* closure of $K_1 + K_2$ is $\overline{K}_1^* + \overline{K}_2^*$. Thus there are u_1 in $\overline{K}_1^*$ and u_2 in $\overline{K}_2^*$ such that $x = u_1 + u_2$. We can choose sequences $\{x_1(n)\}$ in K_1 and $\{x_2(n)\}$ in K_2 such that $\lim_n x_1(n) + x_2(n) = x$. Let $y_1(n) = x_1(n) + u_2$ and $y_2(n) = x_2(n) + u_1$. Then $\{y_1(n)\}$ and $\{y_2(n)\}$ are bounded sequences in $\overline{K}_1^* + \overline{K}_2^*$. Since $\lim_n[y_1(n) + y_2(n)]/2 = x$, we have weak-$\lim_n y_1(n) =$ weak-$\lim_n y_2(n) = x$. Thus by Lemma 4 the sequence $\{x_i(n)\}$ is weakly convergent for $i = 1$ and 2. It follows that $u_i \in K_i$. Therefore, x is in $K_1 + K_2$. Now suppose x is a strongly extreme point of the norm closure of $K_1 + K_2$. By Theorem 3, the element x is also a strongly extreme point of K. Thus x is an extreme point of the weak* closure of K in $X^{(4)}$. Therefore, x is in $K_1 + K_2$. Q.E.D.

Corollary 6. *Let X be a Banach space. The following are equivalent*:

(1) *The space X has the RNP.*

(2) *For any equivalent norm $|\ |$ on X, the unit ball $B_{(X,|\ |)}$ has a strongly extreme point.*

(3) *For any equivalent norm $|\ |$ on X, the unit ball $B_{(X,|\ |)}$ has an extreme point of $B_{(X^{(4)},|\ |)}$.*

Proof. It is obvious that (1) implies (2). By Theorem 3, every strongly extreme point of B_X is an extreme point of $B_{X^{(4)}}$. Thus (2) implies (3). So it remains to show that (3) implies (1). Let $A \subset X$ be nonempty, bounded, and weakly closed. Let $K = \overline{\mathrm{co}}(A \cup -A)$, and let $|\ |$ be the Minkowski functional determined by $K + B_X$ where B_X is the unit ball of X with respect to the original norm. Then $|\ |$ is an equivalent norm on X such that the unit ball $B_{(X,|\ |)}$ is the norm closure of $K + B_X$. Let x be an element in X such that x is an extreme point of the unit ball $B_{(X^{(4)},|\ |)}$ of $X^{(4)}$. By Theorem 5, there are $y \in K$ and $z \in B_X$ such that $x = y + z$. It is obvious that x is an extreme point of $B_{(X^{**},|\ |)}$ and $B_{(X^{**},|\ |)} = \overline{K}^* + B_{X^{**}}$. Thus y is an extreme point of $\overline{K}^*$. By the Krein-Milman Theorem, the set $\operatorname{ext} \overline{K}^*$ is contained in the weak* closure of $A \cup -A$. Since A is weakly closed, we have $y \in A$ or $y \in -A$. In any case the set A has an extreme point. Therefore, X has the RNP [1, Corollary 3.76]. Q.E.D.

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