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ON VOLTERRA EQUATIONS ASSOCIATED WITH A LINEAR OPERATOR

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ABSTRACT. In this work we define the Hille-Yosida space, in the sense of S. Kantorovitz, for a Volterra equation of convolution type.

1. INTRODUCTION

Let X be a Banach space and let A be a closed linear operator with densely defined domain $D(A)$ on X. Let $k \in L^1_{\mathrm{loc}}(R_+)$. In this work we consider the following Volterra equation of convolution type

$$(1.1) \qquad u(t) = f(t) + \int_0^t k(t-s)Au(s)\,ds, \qquad t \in J := [0, T],$$

where $f \in C(J; X)$.

Definition 1.1. A strongly continuous family of bounded and linear operators $\{R(t) : t \geq 0\}$ defined in X, which commutes with A and satisfies the equation

$$(1.2) \qquad R(t)x = x + \int_0^t k(t-s)AR(s)x\,ds, \qquad t \geq 0, \ x \in D(A),$$

is called a *resolvent family* for equation (1.1).

The existence of a resolvent family allows us to find a solution of equation (1.1) by means of the formula.

$$(1.3) \qquad u(t) = R(t)u(0) + \int_0^t R(t-s)f'(s)\,ds, \qquad t \in J,$$

if $f \in W^{1,1}(J, X)$ at least.

The study of diverse properties of resolvent families such as the regularity, positivity, periodicity, approximation, uniform continuity, compactness, and others are studied by several authors under different conditions on the kernel k and the operator A (see, e.g., [3, 6, 11–13, 15] and the references therein).

In [9] Kantorovitz (see also [10]) proved that any linear (unbounded) operator A on a Banach space X such that the resolvent set contains a half line $(0, \infty)$

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generates a strongly continuous semigroup of contractions on a certain subspace Z of X. This so-called Hille-Yosida space is maximal-unique in a suitable sense. Recently, the same problem has been considered in the context of a strongly continuous operator cosine family of contractions by Cioranescu [2] and extended for the case of Fréchet spaces by Henriquez and Hernández [7].

In this work we establish the existence of a "Hille-Yosida Banach space" for resolvent families of equation (1.1). We show that there exists a linear manifold $Z_k \subset X$, depending on the kernel k, and a norm $|\cdot|_k$ majorizing the given norm, such that $(Z_k, |\cdot|_k)$ is a Banach space, and the "restriction of equation (1.1) to Z_k" admits a resolvent family of operators. In particular, the result provides us with a subspace of initial values in X for which the Volterra equation (1.1) has a unique solution given by (1.3). We remark that our result extends those of Kantorovitz and Cioranescu. See also [4] for another closely related generalization of the results in [2, 9].

We will require the following theorem on generation of resolvent families due to Da Prato and Ianelli [3].

Theorem 1.2. *Suppose A is a closed linear densely defined operator in the Banach space X and $k \in L^1_{\text{loc}}(R_+)$ satisfies*

(H_k) $\quad k \in L^1_{\text{loc}}(R_+)$ *has an absolutely convergent Laplace transform*:

$$\hat{k}(\lambda) := \int_0^\infty e^{-\lambda t} k(t)\, dt$$

and is nonzero for every $\operatorname{Re}\lambda > \omega$.

Then there exists a resolvent family $\{R(t)\}_{t\geq 0}$ *for the equation* (1.1) *such that*

$$\|R(t)\| \leq M e^{wt} \quad \text{for all } t \geq 0 \tag{1.4}$$

and some constant $M \geq 1$ *if and only if*

$$1/\hat{k}(\lambda) \in \rho(A) \quad \text{for all } \lambda > \omega \tag{1.5}$$

and

$$\|[(\lambda - \lambda\hat{k}(\lambda)A)^{-1}]^{(n)}\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}} \quad \text{for all } \lambda > \omega, \ , \ n \in N_0 := N \cup \{0\}.$$

Let $\{R(t)\}_{t\geq 0}$ be a resolvent family for equation (1.1). If $w = 0$ and $M = 1$ in (1.4), $R(\cdot)$ is called a *resolvent family of contractions.*

We remark that Theorem 1.2 is essentially Widder's theorem and expresses that the existence of at most exponentially growing resolvent families can be characterized in terms of their Laplace transform, provided the kernel $k(t)$ is Laplace transformable (condition (H_k)). In fact, if $R(t)$ is a resolvent family for (1.1) such that (1.4) holds, then $\widehat{R}(\lambda) = (\lambda - \lambda\hat{k}(\lambda)A)^{-1}$.

2. Volterra equations associated with a linear operator

Throughout this section, X is a Banach space with norm $\|\cdot\|$. Let A be a linear operator and $k \in L^1_{\text{loc}}(R_+)$ such that the condition (H_k) in §1 with $w = 0$ holds.

We assume additionally that

$$1/\hat{k}(\lambda) \in \rho(A) \quad \text{for every } \lambda > 0. \tag{2.1}$$

Let $H(\lambda) \equiv (\lambda - \lambda\hat{k}(\lambda)A)^{-1}$ for $\lambda > 0$, and we define

$$Y_k \equiv \left\{ x \in X : |x|_k := \operatorname*{Sup}_{\substack{\lambda_j > 0 \\ n_j, k \in N_0}} \left\| \prod_{j=1}^{k} \left(\frac{1}{n_j!}\right) \lambda_j^{n_j+1} H^{(n_j)}(\lambda_j)x \right\| < +\infty \right\}$$

where for $k = 0$ the product is defined as x.

It is clear that $|\cdot|_k$ is a norm on Y_k.

Proposition 2.1. $(Y_k, |\cdot|_k)$ *is Banach space.*

Proof. Let $\{x_i\} \subset Y_k$ be a Cauchy sequence. We observe that $\|x\| < |x|_k$ if $x \in X$. Then $\{x_i\}$ is also a Cauchy sequence on X. Let $x \in X$ be the limit of the sequence.

(i) $x \in Y_k$. In fact, let $\lambda_j > 0$, n_j, $k \in N_0$ be fixed. Then

$$\left\| \prod_{j=1}^{k} \left(\frac{1}{n_j!}\right) \lambda_j^{n_j+1} H^{(n_j)}(\lambda_j)x \right\| = \lim_{i\to\infty} \left\| \prod_{j=1}^{k} \left(\frac{1}{n_j!}\right) \lambda_j^{n_j+1} H^{(n_j)}(\lambda_j)x_i \right\|$$
$$\leq \limsup_{i\to\infty} |x_i|_k < +\infty .$$

(ii) $\{x_i\}$ converges to $x \in Y_k$.

Let $\varepsilon > 0$. There exists $N := N(\varepsilon) \in \mathbf{N}$ such that $|x_i - x_l|_k < \varepsilon$ for i, $l > N$. Hence for every $\lambda_j > 0$, k, $n_j \in N_0$,

$$\left\| \prod_{j=1}^{k} \left(\frac{1}{n_j!}\right) \lambda_j^{n_j+1} H^{(n_j)}(\lambda_j)(x_i - x_l) \right\| < \varepsilon \quad \text{for } i, l > N .$$

Making $l \to \infty$ and taking the supremum over all $\lambda_j > 0$, n_j, $k \in N_0$ we obtain

$$|x_i - x|_k < \varepsilon \quad \text{for } i > N . \quad \square$$

Definition 2.2. Let $A_y : D(A_y) \subseteq Y_k \to Y_k$ be defined by $A_y x = Ax$ where

$$D(A_y) := \{x \in D(A) : x, Ax \in Y_k\} .$$

We denote $Z_k := \overline{D(A_y)}$, where the closure is taken in the norm $|\cdot|_k$.

Lemma 2.3. *With the preceding definitions and hypothesis, we have:*

(1) A_y *is a closed linear operator on* Y_k.

(2) $\lambda - \lambda\hat{k}(\lambda)A_y$ *is invertible on* Y_k *for each* $\lambda > 0$.

(3) $\|H_y^{(n)}(\lambda)\|_{B(Y_k)} \leq n!/\lambda^{n+1}$ *for each* $\lambda > 0$ *and* $n \in N_0$ *where* $H_y(\lambda) := (\lambda - \lambda\hat{k}(\lambda)A_y)^{-1}$.

Proof. Let $\lambda > 0$ be fixed. From (2.1) we have the identities

$$(\lambda - \lambda\hat{k}(\lambda)A)H(\lambda)x = x \quad \text{for each } x \in X ,$$
$$H(\lambda)(\lambda - \lambda\hat{k}(\lambda)A)x = x \quad \text{for each } x \in D(A) .$$

Let $y \in Y_k$ be fixed. Certainly $H(\lambda)y \in D(A)$. Moreover

$$|H(\lambda)y|_k = \operatorname{Sup}\left\|\prod_{j=1}^{k}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H^{(n_j)}(\lambda_j)H(\lambda)y\right\|$$
$$\leq \|H(\lambda)\| \operatorname{Sup}\left\|\prod_{j=1}^{k}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H^{(n_j)}(\lambda_j)y\right\| = \|H(\lambda)\|\,|y|_k$$

and we obtain $H(\lambda)y \in Y_k$. On the other hand, from the identity

$$AH(\lambda)y = \frac{1}{\hat{k}(\lambda)}H(\lambda)y - \frac{1}{\lambda\hat{k}(\lambda)}y$$

we have that $AH(\lambda)y \in Y_k$. Therefore $H(\lambda)y \in D(A_y)$, and we conclude that

$$(\lambda - \lambda\hat{k}(\lambda)A_y)H(\lambda)y = y \quad \text{for each } y \in Y_k.$$

Now if $y \in D(A_y)$ then, in particular, $y \in D(A)$ and, therefore,

$$H(\lambda)(\lambda - \lambda\hat{k}(\lambda)A_y)y = y.$$

This proves the assertion (2). In particular, $\rho(A_y) \neq \varnothing$ and hence A_y is a closed linear operator; moreover,

$$H_y(\lambda) = H(\lambda)|Y_k \quad \text{for every } \lambda > 0.$$

Finally, let $y \in Y_k, \lambda > 0$, $n \in N_0$ be fixed. We have

$$\left|\frac{1}{n!}\lambda^{n+1}H_y^{(n)}(\lambda)y\right|_k = \operatorname*{Sup}_{\substack{\lambda_j>0 \\ k,\, n_j\in N_0}}\left\|\prod_{j=1}^{k}\frac{1}{n_j!}\lambda_j^{n_j+1}H^{(n_j)}(\lambda_j)\frac{1}{n!}\lambda^{n+1}H_y^{(n)}(\lambda)y\right\|$$
$$\leq \operatorname{Sup}\left\|\prod_{j=1}^{k+1}\frac{1}{n_j!}\lambda_j^{n_j+1}H^{(n_j)}(\lambda_j)y\right\|$$

where $\lambda_j > 0$ are arbitrary for $1 \leq j \leq k$, $\lambda_{k+1} = \lambda$, and $n_{k+1} := n$.

Therefore

$$\left|\frac{1}{n!}\lambda^{n+1}H_y^{(n)}(\lambda)y\right|_k \leq |y|_k,$$

which proves (3). □

Lemma 2.4. *Define* $A_k\colon D(A_k) \subseteq Z_k \to Z_k$ *as* $A_k x := A_y x$ *for each* $x \in D(A_k)$, *where*

$$D(A_k) := \{x \in D(A_y) : x,\, A_y x \in Z_k\}.$$

Then A_k *is a closed linear operator such that* $\overline{D(A_k)} = Z_k$ *and*

(1) $\lambda - \lambda\hat{k}(\lambda)A_k$ *is an invertible operator in* Z_k *for* $\lambda > 0$,

(2) $\|H_k^{(n)}(\lambda)\|_{B(Z_k)} \leq n!/\lambda^{n+1}$ *for every* $\lambda > 0$, $n \in N_0$ *where* $H_k(\lambda) := (\lambda - \lambda\hat{k}(\lambda)A_k)^{-1}$.

Proof. We observe that $H_k(\lambda) = H_y(\lambda)|Z_k$. Then the result is a direct consequence of [8, Theorem 12.2.4] and Lemma 2.3. □

As a consequence, we obtain the main result of this section on the existence of resolvent families.

Theorem 2.5. *Let A be a linear operator defined in a Banach space X and $k \in L^1_{\mathrm{loc}}(R_+)$ such that condition (H_k) holds. Assume that $(1/\hat{k}(\lambda) - A)^{-1}$ exists for every $\lambda > 0$. Then there exists a linear subspace Z_k and a norm $|\cdot|_k$ such that $(Z_k, |\cdot|_k)$ is a Banach space and the equation*

$$u(t) = f(t) + \int_0^t k(t-s) A_k u(s)\, ds$$

admits a resolvent family of contractions on Z_k.

Proof. According to our hypothesis, we can apply the generation theorem for resolvent families due to Da Prato and Iannelli (see §1, Theorem 1.2) and the result follows from Lemma 2.4. □

The following result shows us that the spaces Z_k are maximal-unique in a certain sense.

Theorem 2.6. *Under the same hypothesis of Theorem 2.5, if $(W_k, \|\cdot\|_k)$ is a Banach space such that $W_k \subset X$, $\|\cdot\| \le \|\cdot\|_k$, and equation (1.1) with $B_k := A|D(B_k)$, $D(B_k) := \{x \in D(A)/x, Ax \in W_k\}$ admits a resolvent family $R_k(t)$, $t \ge 0$, of contractions on W_k, then $W_k \subset Z_k$, $|\cdot|_k \le \|\cdot\|_k$, and $B_k \subset A_k$.*

Proof. The proof follows the same lines of Theorem 3.1 in [2]. We give here the argument for the sake of completeness.

Suppose that $(W_k, \|\cdot\|_k)$, B_k, and $R_k(t)$, $t \ge 0$, are as in the statement of the theorem. Then, because $H(\lambda)$ is the Laplace transform of $R_k(t)$ (see [3]), we have for $x \in W_k$, $\lambda > 0$, and $n \in \mathbf{N}_0$

$$\begin{aligned}
\left\| \frac{1}{n!}\lambda^{n+1} H^{(n)}(\lambda)x \right\| &= \left\| \frac{1}{n!}\lambda^{n+1} \left(\frac{d}{d\lambda}\right)^n \left(\int_0^\infty e^{-\lambda t} R_k(t)x\, dt\right) \right\| \\
&\le \frac{1}{n!}\lambda^{n+1} \int_0^\infty t^n e^{-\lambda t} \|R_k(t)x\|\, dt \\
&\le \frac{1}{n!}\lambda^{n+1} \int_0^\infty t^n e^{-\lambda t} \|R_k(t)x\|_k\, dt \\
&\le \frac{1}{n!}\lambda^{n+1} \int_0^\infty t^n e^{-\lambda t} \|x\|_k\, dt = \|x\|_k .
\end{aligned}$$

We conclude that for $x \in W_k$, $|x|_k \le \|x\|_k$, that is, $W_k \subset Y_k$.
It follows that

$$\begin{aligned}
D(B_k) &:= \{x \in D(A)/x, Ax \in W_k\} \\
&\subset \{x \in D(A)/x, Ax \in Y_k\} =: D(A_y).
\end{aligned}$$

Hence

$$W_k = \overline{D(B_k)}^{\|\cdot\|_k} \subset \overline{D(A_y)}^{\|\cdot\|_k} \subset \overline{D(A_y)}^{|\cdot|_k} = Z_k .$$

Finally, this implies that $D(B_k) \subset D(A_k)$ and $B_k \subset A_k$. □

Taking $k(t) \equiv 1$ or $k(t) \equiv t$ we obtain from Theorem 2.5 the following

Corollary 2.7 [9]. *Let A be a linear operator on X such that $(0, \infty) \subset \rho(A)$. Then there exists a linear subspace $Z_1 \subset X$ and a norm $|\cdot|_1$ such that $(Z_1, |\cdot|_1)$ is a Banach space and the restriction A_1 of A to Z_1 is the infinitesimal generator of a C_0-semigroup of contractions on Z_1.*

Corollary 2.8 [2]. *Let A be a linear operator on X such that $(0, \infty) \subset \rho(A)$. Let A_t be the operator in Z_t defined as above. Then A_t is the infinitesimal generator of a strongly continuous cosine family of contractions on Z_t.*

We can obtain more information about the operators A_k if one assumes a certain regularity on the kernel k as follows.

Proposition 2.9. *Let A be a linear operator defined on X and $k \in L^1_{\mathrm{loc}}(R_+)$ a positive function, which satisfies (H_k) with $\omega = 0$. Assume that $1/\hat{k}(\lambda) \in \rho(A)$ for every $\lambda > 0$. Then A_k generates a strongly continuous semigroup of contractions on Z_k.*

Proof. For the proof of the Hille-Yosida Theorem [14, Theorem 3.1] it is sufficient to have $(\alpha, \infty) \subset \rho(A_k)$ and $\|\lambda(\lambda - A_k)^{-1}\|_{B(Z_k)} \leq 1$ $\forall \lambda > \alpha$, for some real α.

In order to prove this, we take $n = 0$ in Lemma 2.4(2) and obtain

$$\left\| \left(\frac{1}{\hat{k}(\lambda)} - A_k \right)^{-1} \right\|_{B(Z_k)} \leq \hat{k}(\lambda) \quad \text{for every } \lambda > 0. \tag{2.2}$$

Now, because k is positive, the map $\lambda \to 1/\hat{k}(\lambda)$ is positive and increasing. Moreover, condition (H_k) implies that $1/\hat{k}(\lambda)$ tends to infinity as $\lambda \to \infty$. Therefore (2.2) implies the assertion. □

Remark 2.10. As a consequence, under the same hypothesis of Proposition 2.9 on the kernel k and if $(0, \infty) \subset \rho(A)$, it follows from the maximal uniqueness property of Z_1 (Corollary 2.7) that $Z_k \subset Z_1$, $|\cdot|_1 \leq |\cdot|_k$, and $A_k \subset A_1$.

Problem. Find some type of order relation, $\prec$, on the kernels k such that $k_1 \prec k_2$ implies $Z_{k_1} \subset Z_{k_2}$.

Let $\mu \in \mathbf{C}$ and $t \in [0, T]$ be fixed. If $k \in L^2_{\mathrm{loc}}(R_+)$ (see also [1] for other conditions) we can define the function

$$r(t, \mu) = 1 + \int_0^t q(s, \mu)\, ds \tag{2.3}$$

where

$$q(s, \mu) := \sum_{n=1}^{\infty} k^{*n}(s)\mu^n, \qquad s \geq 0.$$

Here we denote by $k^{*n} = k * k * \cdots * k$ the n-times convolution of the kernel k, where by definition,

$$(k * k)(t) = \int_0^t k(t-s)k(s)\, ds.$$

Note that $r(t, \mu)$ satisfies [12, Lemma 2.1]

$$r(t, \mu) = 1 + \mu \int_0^t k(t-s)r(s, \mu)\, ds \tag{2.4}$$

(compare Definition 1.1). Therefore, provided the kernel $k(t)$ is Laplace transformable, we have

$$(\lambda - \lambda\hat{k}(\lambda)\mu)^{-1} = \int_0^{\infty} e^{-\lambda t} r(t, \mu)\, dt$$

(see remark following Theorem 1.2).

We define C_k as the set of all $\mu \in \sigma_p(A)$ such that the map $t \to r(t, \mu)$, from $[0, \infty) \to \mathbf{C}$, is bounded.

Concerning the nontriviality of Z_k we prove:

Proposition 2.11. *Let x be an eigenvector of A corresponding to the eigenvalue $\alpha \in C_k$. Then $x \in Z_k$.*

Proof. Let x be an eigenvector of A with eigenvalue α such that the map $t \to r(t, \alpha)$ is bounded. Let $\lambda > 0$ and $n \in N$ be fixed. Then

$$\begin{aligned}\frac{1}{n!}\lambda^{n+1}H^{(n)}(\lambda)x &= \frac{1}{n!}\lambda^{n+1}\left(\frac{d}{d\lambda}\right)^n(\lambda - \lambda\hat{k}(\lambda)\alpha)^{-1}x \\ &= \frac{1}{n!}\lambda^{n+1}\left(\frac{d}{d\lambda}\right)^n\left(\int_0^\infty e^{-\lambda t}r(t, \alpha)\,dt\right)x.\end{aligned}$$

Therefore

$$\begin{aligned}\left\|\frac{1}{n!}\lambda^{n+1}H^{(n)}(\lambda)x\right\| &\le \frac{1}{n!}\lambda^{n+1}\int_0^\infty t^n e^{-\lambda t}|r(t, \alpha)|\,dt\|x\| \\ &\le \left(\frac{1}{n!}\lambda^{n+1}\int_0^\infty t^n e^{-\lambda t}\,dt\right)\operatorname*{Sup}_{t\ge 0}\|r(t, \alpha)\|\,\|x\|.\end{aligned}$$

This implies that $|x|_k \le \mathrm{Sup}_{t\ge 0}\|r(t, \alpha)\|\,\|x\|$ and, consequently, $x \in D(A_y) \subseteq Z_k$. $\square$

Example 2.12. Let A be a closed linear and densely defined operator on a Banach space X such that

(1) $(0, \infty) \subset \rho(A)$,
(2) $\sigma_p(A) \cap \mathbf{R}_- \neq \varnothing$.

We consider equation (1.1) with singular kernel given by $k(t) := 1/\sqrt{t}$. Then $\hat{k}(\lambda) = \sqrt{\pi/\lambda}$ for $\lambda \neq 0$ and, by application of the Laplace transform to (2.4), we obtain for λ sufficiently large,

$$\hat{r}(\lambda, \mu) = \frac{1}{\lambda(1 - \mu\hat{k}(\lambda))} = \frac{\lambda^{-1/2}}{\lambda^{1/2} - \mu\sqrt{\pi}}.$$

Next, we take the inverse Laplace transform and obtain

$$r(t, \mu) = e^{\mu^2\pi t}\int_{-\mu\sqrt{\pi t}}^\infty e^{-s^2}\,ds, \qquad \mu \in \mathbf{R},\ t \ge 0.$$

It is easy to see that for every $\lambda \le 0$, $0 < r(t, \lambda) < \sqrt{\pi}/2$ for all $t \ge 0$ and

$$C_k = \mathbf{R}_- \cap \sigma_p(A).$$

Then equation (1.1) has a solution defined on a subspace Z_k of X, which contains the span of all eigenvectors corresponding to real, negative eigenvalues of A. Moreover, A_k is the infinitesimal generator of a strongly continuous semigroup of contractions on Z_k.

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