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A PROOF OF THE EXISTENCE OF LEVEL 1 ELLIPTIC COHOMOLOGY

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ABSTRACT. Landweber provided two proofs of the existence of (level 2) elliptic cohomology (Lecture Notes in Math., vol. 1326, Springer-Verlag, New York, 1988, pp. 69–93). As Baker pointed out (J. Pure Appl. Algebra **63** (1990), 1–11), one of these proofs gives a level 1 elliptic cohomology theory as well. In this note we provide an alternative proof of the existence of level 1 elliptic cohomology. The idea here is to use Landweber's direct proof of the existence of level 2 elliptic cohomology and an integrality argument to deduce the existence of level 1 elliptic cohomology from that.

1. INTRODUCTION

Elliptic cohomology was originally defined by Landweber, Ravenel, and Stong [LRS, Land2]. There is a universal elliptic genus

$$\phi\colon \mathrm{MSO}_* \to S$$

that associates to an oriented manifold of dimension $2k$ a modular form over $\mathbf{Z}[\frac{1}{2}]$ of weight k for the congruence subgroup

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) | c \equiv 0 \pmod 2 \right\}.$$

The graded ring of all such modular forms, S, is isomorphic to $\mathbf{Z}[\frac{1}{2}][\delta, \varepsilon]$, where δ, ε have weights 2 and 4 respectively and so occur in grades 4 and 8.

Let $\Delta = 2^{12}\varepsilon(\delta^2 - \varepsilon)^2$. (The reason for the factor 2^{12} will be clear below.) One then forms the tensor product

$$\mathrm{MSO}^*(X) \otimes_{\mathrm{MSO}_*} S[\Delta^{-1}].$$

Here and throughout the paper, we take X to be a finite CW complex, though by working with homology instead we could avoid this assumption. Landweber, Ravenel, and Stong show that this tensor product satisfies the hypotheses of the Landweber exact functor theorem [Land1] and is, therefore, a cohomology theory. Since we will be comparing S to level 1 modular forms where 3 should be inverted as well, let us redefine $S = \mathbf{Z}[\frac{1}{6}][\delta, \varepsilon]$.

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In [Bak1] Baker defined a genus

$$\psi\colon \mathrm{MSO}_* \to R$$

taking values in the graded ring of modular forms over $\mathbf{Z}[\frac{1}{6}]$ for all of $\mathrm{SL}_2(\mathbf{Z})$. The ring R is isomorphic to $\mathbf{Z}[\frac{1}{6}][E_4, E_6]$, where E_4 and E_6 are Eisenstein series of weights 4 and 6, and so are in grades 8 and 12. Let

$$\Delta = \frac{E_4^3 - E_6^2}{2^6 3^3}.$$

Baker then points out that the work of Landweber [Land2] shows that

$$\mathrm{MSO}^*(X) \otimes_{\mathrm{MSO}_*} R[\Delta^{-1}]$$

is a cohomology theory, level 1 elliptic cohomology.

In the present paper, we give another proof of this fact using the relation between level 1 and level 2 modular forms. An outline of the proof is as follows. There is a canonical ring inclusion $\alpha\colon R \to S$. It is shown by Landweber in [Land2] that the formal group laws over S induced by ϕ and $\alpha\psi$ are strictly isomorphic. Thus we know that $\alpha\psi$ satisfies the hypotheses of the Landweber exact functor theorem if Δ is inverted. We then have to show that ψ does. This depends on the integrality of α.

We need a lemma about formal group laws. Let p be a prime. Recall from [Rav] that any formal group law over a $\mathbf{Z}_{(p)}$-algebra A is canonically isomorphic to a p-typical formal group law. Any p-typical formal group law is induced from the universal one by a ring homomorphism $f\colon BP_* \to A$, where $BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$. We use the Araki generators and let $v_0 = p$.

Lemma 1. *Suppose $f, g\colon BP_* \to A$ induce strictly isomorphic formal group laws over A. Then for all n,*

$$f(v_n) \equiv g(v_n) \quad \mathrm{mod}(g(v_0), g(v_1), \dots, g(v_{n-1})).$$

In particular, the ideals generated by

$$(f(v_0), f(v_1), \dots, f(v_n)) \quad \textit{and} \quad (g(v_0), g(v_1), \dots, g(v_n))$$

are the same.

Proof. Let F and G be the formal group laws associated with f and g, and let h be the strict isomorphism between F and G. Recall that the p-series $[p]_F(x)$ is the formal sum of the $v_n x^{p^n}$:

$$[p]_F(x) = \sum\nolimits^F f(v_n)x^{p^n}.$$

A similar statement holds for G. Since h is an isomorphism, we have

$$h([p]_F(x)) = [p]_G(h(x)).$$

Since h is strict, $h(x) \equiv x \,(\mathrm{mod}\, x^2)$. It is now easy to prove the lemma by induction on n. $\square$

Theorem 1 [Bak1]. *Baker's genus $\psi\colon \mathrm{MSO}_* \to R$ induces a cohomology theory*

$$\mathrm{MSO}^*(X) \otimes_{\mathrm{MSO}_*} R[\Delta^{-1}].$$

Proof. As mentioned above, there is a canonical ring inclusion $\alpha\colon R \to S$. It is easy to check using the first few q-expansion coefficients that $\alpha(E_4) =$

$2^6(\delta^2+3\varepsilon)$, $\alpha(E_6) = 2^9\delta(-\delta^2+9\varepsilon)$. One then checks that $\alpha(\Delta) = \Delta$. This is the reason for the factor of 2^{12} in the definition of $\Delta \in S$.

Landweber shows that the formal group laws induced by $\alpha\psi$ and ϕ are strictly isomorphic. In his statement, he considers fields K of characteristic not 2 or 3, but he actually proves it for the universal example as we have here. He also uses the Weierstrass curve $y = 4x^2 - g_2x - g_3$ rather than the Tate curve $y = 4x^3 - (1/12)E_4x + (1/216)E_6$ used by Baker. As Baker explains, this only changes R by an isomorphism and so does not affect the result.

Landweber also shows that ϕ satisfies the hypotheses of the exact functor theorem. That is, for a fixed prime $p > 3$, he shows that $\phi(p) = p$ is a non-zero-divisor, that $\phi(v_1)$ is a non-zero-divisor $\bmod p$, and that $\phi(v_2)$ is a unit $\bmod(p, \phi(v_1))$ when Δ is inverted. In fact, he shows that $\phi(v_2) \equiv \Delta^{(p^2-1)/12}$ $\bmod(p, \phi(v_1))$. By the above lemma, the same facts are true for $\alpha\psi$.

Now we must show that they are true for ψ. It is certainly clear that $\psi(p) = p$ is a non-zero-divisor in $R[\Delta^{-1}]$. It is also easy to see that $\tilde{\alpha}\colon R\otimes \mathbf{F}_p \to S\otimes \mathbf{F}_p$ is injective. Indeed, if $x = py$ and x is a level 1 modular form, so is y. This implies that $\psi(v_1)$ is not a zero-divisor in $R[\Delta^{-1}] \otimes \mathbf{F}_p$. Indeed, suppose $\psi(v_1)x = 0$. Then $\psi(v_1)\tilde{\alpha}x = \tilde{\alpha}\psi(v_1)\tilde{\alpha}x = 0$. Thus $\tilde{\alpha}x = 0$, so $x = 0$.

Now consider $\hat{\alpha}\colon (R\otimes \mathbf{F}_p)/(\psi(v_1)) \to (S\otimes \mathbf{F}_p)/(\tilde{\alpha}\psi(v_1))$. We claim that this map is injective. Indeed, we prove the stronger fact that if $f \neq 0$, $g \in R\otimes \mathbf{F}_p$, and $\tilde{\alpha}(g) = \tilde{\alpha}(f)h$ where $h \in S\otimes \mathbf{F}_p$, then in fact h is in the image of $\tilde{\alpha}$. It clearly suffices to prove this for irreducible f. But in that case (f) is a prime ideal.

It should be well known in the theory of modular forms that S is integral over R, but the only reference I know is [Bak2]. In this case, one can simply see that δ and ε satisfy monic cubic polynomials over R. Indeed,

$$\delta^3 - (3/2^8)\alpha(E_4)\delta + (1/2^{11})\alpha(E_6) = 0$$

and

$$\varepsilon^3 - (1/2^6)\alpha(E_4)\varepsilon^2 + (1/2^{14})\alpha(E_4^2)\varepsilon - (1/2^{14})\alpha(\Delta) = 0.$$

Then $S\otimes \mathbf{F}_p$ is integral over $R\otimes \mathbf{F}_p$ as well.

Thus we can apply the theorem of Cohen and Seidenberg [Jac, p. 411], which says that any prime ideal in $R\otimes \mathbf{F}_p$ is the contraction of a prime ideal P in $S\otimes \mathbf{F}_p$. Thus $(f) = \tilde{\alpha}^{-1}(P)$. In particular, if $\tilde{\alpha}(g) = \tilde{\alpha}(f)h$, then $\tilde{\alpha}(g) \in P$, so $g = fh'$ for some h' in $R\otimes \mathbf{F}_p$. Then $\tilde{\alpha}(h') = h$.

In particular, $\hat{\alpha}$ is injective. The image under $\hat{\alpha}$ of $\psi(v_2)$ is $\Delta^{(p^2-1)/12}$; therefore, we must have $\psi(v_2) = \Delta^{(p^2-1)/12}$. Thus ψ satisfies the hypotheses of the Landweber exact functor theorem and so induces a cohomology theory. □

References

[Bak1] A. Baker, *Hecke operators as operations in elliptic cohomology*, J. Pure Appl. Algebra **63** (1990), 1–11.

[Bak2] ——, *Elliptic genera of level N and elliptic cohomology*, preprint, 1991.

[Jac] N. Jacobson, *Basic algebra*. II, Freeman, San Francisco, CA, 1980.

[Land1] P. Landweber, *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*, Amer. J. Math. **98** (1976), 591–610.

[Land2] ——, *Supersingular elliptic curves and congruences for Legendre polynomials*, Lecture Notes in Math., vol. 1326, Springer-Verlag, New York, 1988, pp. 69–93.

[LRS] P. Landweber, D. Ravenel, and R. Stong, *Periodic cohomology theories defined by elliptic curves* (to appear).

[Rav] D. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, New York, 1986.

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