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TYPICAL INTERSECTIONS OF CONTINUOUS FUNCTIONS WITH MONOTONE FUNCTIONS

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ABSTRACT. For each parameter Φ a typical continuous function intersects every monotone function in a (Φ)-uniformly symmetrically porous set.

1. INTRODUCTION AND NOTATION

This paper generalizes the results of Humke and Laczkovich [1]. The notion of "bilaterally strongly Φ-porosity" in [1] is replaced by the stronger one of "(Φ)-uniformly symmetric porosity" (see Definition 2 and Theorem 3). The main result of the paper, Theorem 4, is obtained using an adaptation of the Banach-Mazur game (Theorem 6), and there is a different approach than that used in [1].

Definition 1. Let $\Phi\colon (0, 1) \to (0, 1]$ be a continuous function. A set $E \subset \mathbb{R}$ is said to be bilaterally strongly Φ-porous if for every $x \in E$ there are sequences of intervals $I_n \subset (x - 1/n, x) \setminus E$ and $J_n \subset (x, x + 1/n) \setminus E$ such that

$$\text{(a)} \qquad \lim_{x\to\infty} \frac{\operatorname{dist}(x, I_n)}{\Phi(|I_n|)} = \lim_{x\to\infty} \frac{\operatorname{dist}(x, J_n)}{\Phi(|J_n|)} = 0.$$

In [1] Humke and Laczkovich proved the following

Theorem 1. *Let $\Phi\colon (0, 1] \to (0, 1]$ be a continuous function. Then a typical continuous function intersects every monotone function in a bilaterally strongly Φ-porous set.*

Notation 1. The family of all continuous increasing functions Φ on $[0, 1]$ for which $\Phi(0) = 0$ will be denoted by G; such functions will be referred to as porosity indices.

Notation 2. Let $\Phi \in G$ and $k \in \mathbb{N}$. By $R(\Phi, k)$ we will denote the set of all $E \subset \mathbb{R}$ for which there are numbers a_k, b_k such that for all $x \in E$ the following hold:

(i) $0 < a_k < b_k < k^{-1}$,

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(ii) $\Phi(b_k - a_k) > a_k$, and
(iii) $[x - b_k\,,\ x - a_k] \cap E = [x + a_k\,,\ x + b_k] \cap E = \varnothing$.

Further let us denote $R(\Phi) = \bigcap_{k=1}^{\infty} R(\Phi, k)$.

Definition 2. Let $\Phi \in G$. Those sets $E \in R(\Phi)$ are said to be (Φ)-uniformly symmetrically porous.

For our purposes Theorem 1 will be slightly reformulated.

Definition 3. Let $\Psi \in G$. We call a set $E \subset \mathbb{R}$ (Ψ)-bilaterally strongly porous if for every $x \in E$ there are sequences of intervals $I_n \subset (x - 1/n\,, x) \setminus E$ and $J_n \subset (x\,, x + 1/n) \setminus E$ such that for every $n \in \mathbb{N}$ both $\operatorname{dist}(x\,, I_n) < \Psi(|I_n|)$ and $\operatorname{dist}(x\,, J_n) < \Psi(|J_n|)$.

We reformulate Theorem 1 as

Theorem 2. *Let $\Psi \in G$. Then the typical continuous function intersects every monotone function in a (Ψ)-bilaterally strongly porous set.*

In fact, for each continuous function Φ: $(0, 1] \to (0, 1]$ there exists $\Psi_0 \in G$ such that $\Psi_0(x) \le \Phi(X)$ for $x \in (0, 1]$. Let us set $\Psi(x) = x\Psi_0(x)$. For $\Psi \in G$ we can find (according to Theorem 2) sequences of intervals $\{I_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, $\operatorname{dist}(x\,, I_n) < \Psi(|I_n|)$ and $\operatorname{dist}(x\,, J_n) < \Psi(|J_n|)$. Then $\operatorname{dist}(x\,, I_n) < \Psi(|I_n|) = |I_n|\Psi_0(|I_n|) \le n^{-1}\Phi(|I_n|)$. Similarly $n \operatorname{dist}(x\,, J_n) < \Phi(|J_n|)$ and, hence, (a).

Therefore Theorem 1 is a consequence of Theorem 2. The fact that Theorem 2 is a consequence of Theorem 1 is evident.

2. Proof of Theorem 3

In this section we will show by Theorem 3 that our main result (Theorem 4) is stronger than the result of Humke and Laczkovich (Theorem 2).

Theorem 3. *Let Φ be the identity function on $[0, 1]$. Then for every $\Psi \in G$ there is $E \subset [0, 1]$ such that E is (Ψ)-bilaterally strongly porous and is not (Φ)-uniformly symmetrically porous.*

Proof. Let $\Psi \in G$. The subsequence $\{p_{k_n}\}_{n=1}^{\infty}$ of the sequence $\{p_k\}_{k=1}^{\infty} = \{(\frac{2}{3})^{k+1}\}_{k=1}^{\infty}$ will be defined by induction; set $k_1 = 1$ and suppose $k_1, k_2, \ldots, k_n$ are given. Since $\Psi \in G$, there exists $k_{n+1} \in \mathbb{N}$ such that $\Psi(p_{k_n} - p_{k_{n+1}}) > p_{k_{n+1}}$.

Now denote

$$\begin{aligned} E = {} & \{0.5 + p_m : k_{2n-1} \le m \le k_{2n}\,,\ n \in \mathbb{N}\} \cup \{0.5\} \\ & \cup \{0.5 - p_m : k_{2n} \le m \le k_{2n+1}\,,\ n \in \mathbb{N}\}. \end{aligned}$$

Obviously there is only one point, namely, 0.5, for which it is necessary to verify (Ψ)-bilaterally strongly porosity. Setting

$$I_n = (0.5 - p_{k_{2n-1}}\,,\ 0.5 - p_{k_{2n}})$$

and

$$J_n = (0.5 + p_{k_{2n+1}}\,,\ 0.5 + p_{k_{2n}}),$$

it is easy to see that the set E is (Ψ)-bilaterally strongly porous. Further for each $a \in (0, 0.2)$ either $[0.5 - 2a,\ 0.5 - a] \cap E \ne \varnothing$ or $[0.5 + a,\ 0.5 + 2a] \cap E \ne \varnothing$. Thus the set E is not (Φ)-uniformly symmetrically porous. Q.E.D.

In Theorem 3 the set E could have been chosen as

$$\begin{aligned} E = &\{0.25 - p_m : k_{2n-1} \leq m \leq k_{2n},\ n \in \mathbb{N}\} \\ &\cup \{0.25 + p_m : k_{2n-1} \leq m \leq k_{2n},\ n \in \mathbb{N}\} \\ &\cup \{0.75 - p_m : k_{2n} \leq m \leq k_{2n+1},\ n \in \mathbb{N}\} \\ &\cup \{0.75 + p_m : k_{2n} \leq m \leq k_{2n+1},\ n \in \mathbb{N}\} \\ &\cup \{0.25, 0.75\}. \end{aligned}$$

Now for each $x \in E$ and each $k \in \mathbb{N}$ there are numbers a_k, b_k such that (i), (ii), and (iii) of Notation 1 are fulfilled, but the uniformity is not preserved, i.e., there are no sequences $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ such that for each $x \in E$ (in our case we put $x = 0.25$ or $x = 0.75$) (i), (ii), and (iii) hold.

The main theorem of this paper reads as follows.

Theorem 4. *Let $\Phi \in G$. Then the typical continuous function intersects each monotone function in a (Φ)-uniformly symmetrically porous set.*

It is clear that Theorem 2 is a consequence of Theorem 4, but by Theorem 3 we showed that Theorem 2 is not a consequence of Theorem 4.

3. Proof of Theorem 2

Notation 3. Let f be a real function, $g \in C[0, 1]$, $x \in \mathbb{R}$, $\varepsilon > 0$. We denote

$$U(x, \varepsilon) = \{y \in \mathbb{R}: |x - y| < \varepsilon\},$$
$$U(g, \varepsilon) = \{\varphi \in C[0, 1] : |\varphi(x) - g(x)| < \varepsilon \text{ for } x \in [0, 1]\},$$
$$M_{f,\varepsilon} = \{(x, y) \in \mathbb{R}^2 : |f(x) - y| < \varepsilon\},$$

and

$$\operatorname{gr} f = \{(x, f(x)) \in \mathbb{R}^2 : x \in [0, 1]\}.$$

For a set $M \subset \mathbb{R}^2$ we denote $P(M) = \{x \in \mathbb{R} :$ there exists $y \in \mathbb{R}$ such that $(x, y) \in M\}$.

We need two lemmas. The first of these is easy to see and is not proved.

Lemma 1. *Let $\delta > 0$, $\alpha > 0$. Let r be a continuous piecewise linear function on interval I, for which $r'_+(x) < -\alpha$, for all $x \in \operatorname{int} I$. Let f be a nondecreasing function. Then there exists an interval of length $2\delta/\alpha$ which contains the set $P(M_{r,\delta} \cap \operatorname{gr} f)$.*

Lemma 2. *Let $U \subset C[0, 1]$ be open and nonempty. Then there exists $n \in \mathbb{N}$ such that for an arbitrary $\gamma > 0$ there is a function $s \in C[0, 1]$ and a number $\delta > 0$ such that for each nondecreasing function f there are intervals $J_1, J_2, \dots, J_n$ for which*

(i) *$|J_i| < \gamma$ for $i = 1, 2, \dots, n$,*
(ii) *$U(s, \delta) \subset U$, and*
(iiii) *$P(M_{s,\delta} \cap \operatorname{gr} f) \subset \bigcup_{i=1}^{n} J_i$.*

Proof. Obviously there exists a continuous and piecewise linear function $h \in C[0, 1]$ and a number $\varepsilon > 0$ such that $U(h, \varepsilon) \subset U$. Denote

$$t = \max\{|h'_+(x)| : x \in (0, 1)\}, \tag{1}$$

choose $n_0 \in \mathbb{N}$ such that

$$n_0 > \frac{t+1}{\varepsilon}, \tag{2}$$

and denote $n = 2n_0$. Further let $\gamma > 0$, $b = \min\{1/n, \gamma/2\}$, and $I = [b, 1/n_0]$. Let u_0 be the continuous and piecewise linear function defined as follows:

(a) u_0 is periodic with the period n_0^{-1};
(b) $u_0(0) = -\frac{\varepsilon}{2}$, $u_0(b) = \frac{\varepsilon}{2}$, $u_0(n_0^{-1}) = -\frac{\varepsilon}{2}$; and
(c) u_0 is linear on $[0, b]$ and on I.

Let us denote $u(x) = u_0(x)_{|[0,1]}$, and let $\delta > 0$ be such that

$$2\delta < \min\{\gamma, \varepsilon\}. \tag{3}$$

Suppose f is nondecreasing. Denote $s = h+u$ and $J_{i+1} = [in_0^{-1}, in_0^{-1}+b]$ for $i = 0, 1, \dots, n_0-1$. The upcoming construction of intervals $J_{n_0+1}, J_{n_0+1}, \dots, J_n$ will use Lemma 1.

From (2) and the definition of the function $u(x)$ it follows that $u'(x) = -\varepsilon/(n_0^{-1} - b) < -\varepsilon n_0 < -t-1$ for $x \in \operatorname{int} I$. Thus by (1)

$$s'_+(x) = u'_+(x) + h'_+(x) < -t - 1 + t = -1$$

for $x \in \operatorname{int} I$. Denote $s_1 = s_{|I}$. From (3) and the fact that $s'_+(x) < -1$ for $x \in \operatorname{Int} I$ it follows from Lemma 1 that there is an interval J_{n_0+1} of the length less than γ for which $P(M_{s_1,\delta} \cap \operatorname{gr} f) \subset J_{n_0+1}$. Similarly we define the intervals J_{n_0+i} for $i = 2, 3, \dots, n_0$. Thus for $s_i = s_{|[(i-1)/n_0+b,\ 1/n_0]}$ it follows that $P(M_{s_i}, \delta \cap \operatorname{gr} f) \subset J_{n_0+i}$. Conclusions (i) and (iii) follow directly from the definition of the intervals $J_1, \dots, J_n$. From (3) and the inequality $|u(x)| \leq \varepsilon/2$ it follows that $U(s, \delta) \subset U(h, \varepsilon)$; but then $U(h, \varepsilon) \subset U$ and conclusion (ii) follows. Q.E.D.

To prove Theorem 4 we introduce the Banach-Mazur game: Assume that X is a complete metric space and $B \subset X$. The Banach-Mazur game is played by two players, (A) and (B). In the first step, (A) chooses an open and nonempty set $U_1 \subset X$, and (B) chooses an open and nonempty set $V_1 \subset U_1$. In the nth step (A) chooses an open and nonempty set $U_n \subset V_{n-1}$ and (B) chooses an open and nonempty set $V_n \subset U_n$. This defines a nonincreasing sequence of open sets. If $\bigcap_{i=1}^{\infty} V \subset B$ then (B) wins. In the opposite case (A) wins.

Theorem 5 (see [1]). *In the Banach-Mazur game there is a winning strategy for the player* (B) *if and only if the set* B *is residual in* X.

Further the Banach-Mazur game will be looked at with respect to the space $X = C[0, 1]$.

Definition 4. A nonempty family P of subsets of $\mathbb{R}$ is called a family of small sets if the relation $A \in P$ and $B \subset A$ yields $B \in P$.

Notation 4. Let P be a family of small sets. The Banach-Mazur game for which $B = \{\varphi \in C[0, 1] : P(\operatorname{gr} f \cap \operatorname{gr} \varphi) \in P$ for every nondecreasing function $f\}$ will be denoted by $\mathrm{BM}(P)$.

Further we will use the $\mathrm{F}(P)$ game, which is described as follows: The $\mathrm{F}(P)$ game is played by two players, (A) and (B). In the first step, (A) chooses a number $n_1 \in \mathbb{N}$ and (B) chooses a real positive number $\gamma_1 > 0$. In the kth step (A) chooses a number $n_k \in \mathbb{N}$ and (B) chooses a real positive number $\gamma_k > 0$. This defines a sequence $n_1, \gamma_1, n_2, \gamma_2, \dots$. If for every sequence $\{T_k\}_{k=1}^{\infty}$ of sets $T_k = \bigcup_{i=1}^{n_k} I_i^k$, where I_i^k $(i = 1, 2, \dots, n_k)$ are intervals shorter than γ_k, $\bigcap_{k=1}^{\infty} T_k \in P$ holds, then (B) wins. In the opposite case (A) wins.

Lemma 3. *Let P be a family of small sets. If there is a winning strategy for the player* (B) *in the* F(P) *game then there is in the* BM(P) *game as well.*

Proof. Suppose that the BM(P) game up to the kth step of (A) is given by the sequence $U_1 \supset V_1 \supset \cdots \supset U_k$, and suppose that the F(P) game up to the $(k-1)$th step of (B) is given by the sequence $n_1, \gamma_1, \dots, n_{k-1}, \gamma_{k-1}$ and that (B) has used a winning strategy. Now for $U_k \subset C[0, 1]$ by Lemma 2 we find $n_k \in \mathbb{N}$. Then with respect to the winning strategy of (B) in the F(P) game for $n_k \in \mathbb{N}$ we obtain a number $\gamma_k > 0$. For U_k, n_k, and γ_k by Lemma 2 we find a function $s \in C[0, 1]$ and a number $\delta_k > 0$. Now put $V_k = U(s_k, \delta_k)$ as the kth step of (B). From Lemma 2 we have $V_k \subset U_k$.

We are going to show that (B) wins in this BM(P) game. If $\bigcap_{i=1}^{\infty} V_i = \varnothing$ there is nothing to be proved. Let $s \in \bigcap_{i=1}^{\infty} V_i$ and let f be a nondecreasing function. Denote $E = P(\operatorname{gr} f \cap \operatorname{gr} s)$. According to Lemma 2 for each $k \in \mathbb{N}$ there are intervals $I_1^k, I_2^k, \dots, I_{n_k}^k$ such that $|I_1^k| < \gamma_k$, $i = 1, 2, \dots, n_k$, and

$$P(\operatorname{gr} f \cap M_{s_k, \delta_k}) \subset T_k \quad \text{where } T_k = \bigcup_{i=1}^{n_k} I_i^k. \tag{4}$$

Obviously $E \subset P(\operatorname{gr} f \cap M_{s_k, \delta_k})$, and regarding (4) we have

$$E \subset \bigcap_{k=1}^{\infty} T_k. \tag{5}$$

Since (B) used the winning strategy in the F(P) game, it follows that $\bigcap_{i=1}^{\infty} T_k \in P$. From (5) and the fact that P is a family of small sets we get $E \in P$. Hence (B) wins in the BM(P) game as well. Q.E.D.

Notation 5. Let P be a family of small sets. Then we denote

$$\begin{aligned} W_P &= \{\varphi \in C[0, 1] : P(\operatorname{gr} f \cap \operatorname{gr} \varphi) \in P \text{ for each } f \text{ monotone}\}, \\ W_P^- &= \{\varphi \in C[0, 1] : P(\operatorname{gr} f \cap \operatorname{gr} \varphi) \in P \text{ for each } f \text{ nonincreasing}\}, \\ W_P^+ &= \{\varphi \in C[0, 1] : P(\operatorname{gr} f \cap \operatorname{gr} \varphi) \in P \text{ for each } f \text{ nondecreasing}\}. \end{aligned}$$

Lemma 4. *Let P be a family of small sets. The set W_P^+ is residual if and only if the set W_P^- is residual.*

Proof. Obviously it is enough to find a homeomorphism H from $C[0, 1]$ on to $C[0, 1]$ such that $H(W_P^+) = W_P^-$. Define $H(\varphi) = -\varphi$ for $\varphi \in C[0, 1]$. Let $\varphi \in W_P^+$, and let f be nonincreasing. Then $P(\operatorname{gr} f \cap \operatorname{gr} -\varphi) = P(\operatorname{gr} -f \cap \operatorname{gr} \varphi) \in P$. Thus $-\varphi \in W_P^-$. Similarly $\varphi \in W_P^-$ yields $-\varphi \in W_P^+$. Q.E.D.

Lemma 5. *Let P be a family of small sets such that the set W_P^+ is residual. Then W_P is also residual in $C[0, 1]$.*

Proof. According to Lemma 4 the set $W_P = W_P^+ \cap W_P^-$ is residual. Because $W_P = W_P^+ \cap W_P^-$, W_P is thus residual also. Q.E.D.

Theorem 6. *Let P be a family of small sets. If there is a winning strategy for the player* (B) *in the* F(P) *game then the set W_P is residual in* $C[0, 1]$.

Proof. According to Lemma 3 there is a winning strategy for (B) also in the BM(P) game, and with respect to Theorem 5 the set W_P^+ is residual. Theorem 6 now follows from Lemma 5. Q.E.D.

Proof of Theorem 4. Let $\Phi \in G$. We need to show that $W_{R(\Phi)}$ is residual. To do that it is enough to verify the assumptions of Theorem 5. Obviously it is enough to find the winning strategy for player (B) in the F($R(\Phi)$) game. Let the kth step of player (A) be given by the number $n_k \in \mathbb{N}$. add We are going to find a number $\gamma_k > 0$ as a kth step of player (B). The set $A = \{(x_1, x_2, \dots, x_{n_k}) \in \mathbb{R}^{n_k} : 0 \le x_1 \le x_2 \le \cdots \le x_{n_k} \le 1\}$, as a subset of the metric space $(\mathbb{R}^{n_k}, \rho)$ with maxim metric ρ (i.e., $\rho(x, y) = \max\{|x_i - y_i| : i = 1, 2, \dots, n_k\}$ where $x = (x_1, x_2, \dots, x_{n_k})$ and $y = (y_1, y_2, \dots, y_{n_k})$), is compact. For $x = (x_1, x_2, \dots, x_{n_k}) \in A$ we define function $F(x) = \sup\{\varepsilon > 0 : \bigcup_{i=1}^{n_k} U(x_1, \varepsilon) \in R(\Phi, k)\}$. Further we will prove that there is a number $\gamma_k > 0$ such that

$$F(x) > \gamma_k \quad \text{for all } x \in A. \tag{6}$$

Because A is compact, in order to prove (6) it is sufficient to prove the following two assertions:

$$F(x) > 0 \quad \text{for all } x \in A, \tag{7}$$

and

$$F \text{ is continuous on } A. \tag{8}$$

The proof of (7) will be divided into two cases.

I. If $x_1 = x_{n_k}$, denote $b_k = 1/(k+1)$. Because $\Phi \in G$, there exists $a_k > 0$ such that $\Phi(b_k - a_k) > a_k$. For $\varepsilon = 2^{-1}a_k$ obviously $\bigcup_{i=1}^{n_k} U(x_1, \varepsilon) \in R(\Phi, k)$.

II. If $x_1 \ne x_{n_k}$, denote $d = \min\{|x_i - x_j| > 0 : i, j = 1, 2, \dots, n_k\}$ and $b_k = \min\{d/2, 1/(k+1)\}$. It is clear that there is a number $a_k > 0$ such that $\Phi(b_k - a_k) > a_k$. For $\varepsilon = 2^{-1}a_k$ it is $\bigcup_{i=1}^{n_k} U(x_i, \varepsilon) \in R(\Phi, k)$.

To prove (8) it is sufficient to show

Lemma 6. *Let $x = (x_1, x_2, \dots, x_{n_k}) \in A$, $y = (y_1, y_2, \dots, y_{n_k}) \in A$, and $\rho(x, y) < \delta$. Then* $F(x) + \delta \ge F(y)$.

Proof. If $F(y) \le \delta$ then clearly Lemma 6 holds. If $F(y) > \delta$ then there exists $\varepsilon > \delta$ such that $\bigcup_{i=1}^{n_k} U(y_i, \varepsilon) \in R(\Phi, k)$. From $\rho(x, y) < \delta$ we have $\bigcup_{i=1}^{n_k} U(x_i, \varepsilon - \delta) \subset \bigcup_{i=1}^{n_k} U(y_i, \varepsilon)$, and since $R(\Phi, k)$ is a family of small sets, we obtain $\bigcup_{i=1}^{n_k} U(x_i, \varepsilon - \delta) \in R(\Phi, k)$. Q.E.D.

To finish the proof of Theorem 4 we will show that in the described game given by $n_1, \gamma_1, n_2, \gamma_2, \dots$ (B) wins.

Let there be a sequence $\{T_k\}_{k=1}^{\infty}$, $T_k = \bigcup_{i=1}^{n_k} I_i^k$, where intervals I_i^k $(i = 1, 2, \dots, n_k)$ satisfy $|I_i^k| < \delta_k$. Then there exists $x = (x_1, x_2, \dots, x_{n_k}) \in A$ such that $T_k \subset \bigcup_{i=1}^{n_k} U(x_i, \gamma_k)$. With respect to (6) and to the fact that $R(\Phi, k)$ is a family of small sets we obtain $T_k \in R(\Phi, k)$. Hence $\bigcap_{i=1}^{\infty} T_i \in R(\Phi, k)$, and also $\bigcap_{i=1}^{\infty} T_i \in \bigcap_{i=1}^{\infty} R(\Phi, i) = R(\Phi)$. Q.E.D.

It is not very difficult to show that Theorem 4 yields the following assertion.

Let $\Phi \in G$. Then a typical continuous function intersects every Lipschitz function in (Φ)-uniformly symmetrically porous set.

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REFERENCES

1. P. D. Humke and M. Laczkovich, *Typical continuous functions are virtually nonmonotone*, Proc. Amer. Math. Soc. **94** (1985), 244–248.
2. ——, *Contributions to the theory of games*, vol. 3, Ann. Math. Stud., no. 39, Princeton Univ. Press, Princeton, NJ, 1957, pp. 159–163.

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