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THE HAUSDORFF DIMENSION OF SELF-SIMILAR SETS UNDER A PINCHING CONDITION

XIAOPING GU

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ABSTRACT. We study self-similar sets in the case where the construction diffeomorphisms are not necessarily conformal. Using topological pressure we give an upper estimate of the Hausdorff dimension, when the construction diffeomorphisms are $C^{1+\kappa}$ and satisfy a κ-pinching condition for some $\kappa \leq 1$. Moreover, if the construction diffeomorphisms also satisfy the disjoint open set condition we then give a lower bound for the Hausdorff dimension.

1. INTRODUCTION

The construction of a self-similar set starts with a $k \times k$ matrix $A = (a_{ij})$ which has entries zeros and ones, with all entries of A^N positive for some $N > 0$; see [H]. For each nonzero a_{ij} we give a contraction map $\varphi_{ij}\colon R^l \to R^l$ with $\|\varphi_{ij}(x) - \varphi_{ij}(y)\| \leq c\|x - y\|$, where $c < 1$ is a constant and we are using the Euclidean norm on R^l. Define the Hausdorff metric by

$$d(E, F) = \inf\{\delta \mid d(x, F) \leq \delta \text{ for all } x \in E, \text{ and } d(y, E) \leq \delta \text{ for all } y \in F\}$$

in the space $\mathscr{C}$ of all nonempty compact subsets of R^l. See, for example, [H] or [F]. The map Φ on the k-fold product space $\mathscr{C}^k$ given by

$$\Phi(F_1, \dots, F_l) = \left(\bigcup_{j=1}^{k} \varphi_{1j}(F_j), \dots, \bigcup_{j=1}^{k} \varphi_{kj}(F_j) \right)$$

is a contraction map. By the Banach Fixed Point Theorem the contraction map Φ has a unique fixed point in $\mathscr{C}^k$, i.e., a vector of compact nonempty subsets of R^l, $(E_1, \dots, E_k) \in \mathscr{C}^k$, with $\bigcup_{a_{ij}=1} \varphi_{ij}(E_j) = E_i$. The union $E = \bigcup_{i=1}^{k} E_i$ is called a self-similar set.

Let $\Sigma = \Sigma_A^+ = \{(x_0, x_1, \dots, x_n, \dots) \mid 1 \leq x_i \leq k \text{ and } a_{x_i x_{i+1}} = 1 \text{ for all } i \geq 0\}$ be the shift space with the following metric: for $\mathbf{x} = (x_0, x_1, \dots)$, $\mathbf{y} = (y_0, y_1, \dots)$ in Σ, $d(\mathbf{x}, \mathbf{y}) = 2^{-n}$ if and only if $n = \min\{m \mid x_m \neq y_m\}$. Let σ be the shift map of Σ, and let $\pi\colon \Sigma \to E$ be given by

$$\pi(x_0, x_1, \dots, x_n, \dots) = \text{the only point in } \bigcap_{n \geq 1} \varphi_{x_0 x_1} \varphi_{x_1 x_2} \cdots \varphi_{x_n x_{n+1}}(E_{x_{n+1}}).$$

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It is clear that π is a Hölder continuous surjective map. We will denote the composition $\varphi_{x_0x_1}\cdots\varphi_{x_{n-1}x_n}$ by $\varphi_{x_0\cdots x_n}$. Also, we assume all φ_{ij} to be C^1 diffeomorphisms and denote the derivative of φ_{ij} at a point x by $T_x\varphi_{ij}$ or $T\varphi_{ij}(x)$.

Definition 1.1. The jth Lyapunov number of a linear map L, denoted by $\alpha_j(L)$, is the square root of the jth largest eigenvalue of LL^*, where L^* is the conjugate of L. Write $\omega_t(L) = \alpha_1(L)\cdots\alpha_{[t]}(L)_{\alpha_{[t]+1}}(L)^{t-[t]}$. For a set of construction diffeomorphisms $\{\varphi_{ij}\}$, define $\lambda_t\colon \Sigma\to R$ for each $t\in[0, l]$ and $\mathbf{x} = (x_0x_1\cdots)\in\Sigma$ by

$$\begin{aligned}\lambda_t(\mathbf{x}) = {}& \log\alpha_1(T\varphi_{x_0x_1}(\pi\sigma\mathbf{x})) + \cdots + \log\alpha_{[t]}(T\varphi_{x_0x_1}(\pi\sigma\mathbf{x}))\\ &+ (t-[t])\log\alpha_{[t]+1}(T\varphi_{x_0x_1}(\pi\sigma\mathbf{x}))\\ = {}& \log\omega_t(T\varphi_{x_0x_1}(\pi\sigma\mathbf{x})).\end{aligned}$$

Here "log" is the natural logarithm.

The constructions and dimensions of self-similar sets have been studied by several authors under various restrictions. In this paper we relax the restrictions on construction diffeomorphisms to a κ-pinching condition, which is defined as follows.

Definition 1.2. We say that a C^1 homeomorphism φ_{ij} satisfies the κ-pinching condition if for all $x\in E$ the derivatives satisfy $\|T_x\varphi_{ij}\|^{1+\kappa}\cdot\|T_{\varphi_{ij}(x)}\varphi_{ij}^{-1}\| < 1$.

Remark. If $T_x\varphi_{ij}T_x\varphi_{ij}^*$ has eigenvalues $\alpha_{1,ij}(x)^2 \geq \cdots \geq \alpha_{l,ij}(x)^2$ where $T_x\varphi_{ij}^*$ denotes the conjugate of $T_x\varphi_{ij}$, the numbers $\alpha_{1,ij}(x), \ldots, \alpha_{l,ij}(x)$ are Lyapunov numbers with $1 > \alpha_{1,ij}(x) \geq \cdots \geq \alpha_{l,ij}(x) > 0$. The pinching condition is equivalent to $\alpha_{1,ij}(x)^{1+\kappa} < \alpha_{l,ij}(x)$.

For the definition and properties of Hausdorff dimension, refer to [K]. Also, we use the definitions and notions of [W] in the discussion concerning topological pressure.

Theorem 1. *Let $\{\varphi_{ij}\}$ be the C^1 construction diffeomorphisms for the self-similar set E, satisfying the κ-pinching condition for some positive number $\kappa\leq 1$. Suppose the derivatives of all $\{\varphi_{ij}\}$ are Hölder continuous of order κ. If t is the unique positive number such that the topological pressure $P(\sigma, \lambda_t) = 0$, then the Hausdorff dimension* $\mathrm{HD}(E)\leq t$.

Let us recall the disjoint open set condition on the construction of self-similar sets; see [H]. It states that for each integer i from 1 to k there is a nonempty open set U_i such that

$$\bigcup_{a_{ij}=1}\varphi_{ij}(U_j)\subset U_i \quad\text{and}\quad \varphi_{ij}(U_j)\cap\varphi_{ik}(U_k) = \varnothing \qquad \text{if } j\neq k.$$

For $n\geq 0$, denote $U_n(\mathbf{x}) = \varphi_{x_0x_1}\varphi_{x_1x_2}\cdots\varphi_{x_{n-1}x_n}(U_{x_n})$. It follows that $E_i\subset\overline{U}_i$ and that the collection $\{U_n(\mathbf{x}) : \mathbf{x}\in\Sigma\}$ is pairwisely disjoint for each fixed n.

Theorem 2. *Let $\{\varphi_{ij}\}$ be the C^1 construction diffeomorphisms for the self-similar set E, satisfying both the κ-pinching condition for some positive number $\kappa\leq 1$, and the disjoint open set condition. Suppose the derivatives of all $\{\varphi_{ij}\}$ are Hölder continuous of order κ. If t is the unique positive number* *such that the topological pressure* $P(\sigma, \lambda_t) = 0$, *then the Hausdorff dimension* $\mathrm{HD}(E) \geq t/(1+\kappa) - l\kappa$.

Remark. We call a C^1 diffeomorphism $C^{1+\kappa}$ if its derivative is Hölder continuous of order κ. If we fix the construction to be $C^{1+\beta}$ for some $\beta > 0$ but let $\kappa \to 0$ for the κ-pinching condition, then our upper and lower bounds will coincide with the estimate for conformal cases in [B1].

Theorem 1 is proved in §2, and Theorem 2 is proved in §3. As a corollary of Theorems 1 and 2, in §4 we will also discuss some continuity in the C^1 topology of the Hausdorff dimension at conformal $C^{1+\kappa}$ constructions under disjoint open set condition. For discussions of the constructions of self-similar sets using similitudes and their dimensions, see [H] and [MW]. For the constructions using "conformal" contraction maps, see [B1]. Other related works can be found in [D, B2, F]. A similar result for basic sets in two dimensions can be found in [MM].

2. The upper bound

Lemma 2.1. *If all construction diffeomorphisms* φ_{ij} *are* $C^{1+\kappa}$ *and satisfy the* κ*-pinching condition, then for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *depending only on* ε, *such that for all* $x \in E$, *all* a *with* $0 < a < \delta$, *and all* $\mathbf{x} = (x_0, x_1, \ldots)$ *in* Σ, *all integers* $n > 0$, *we have*

$$\varphi_{x_0\cdots x_n} B(x, a) \subset \varphi_{x_0\cdots x_n}(x) + (1+\varepsilon)^n T_x \varphi_{x_0\cdots x_n} B(0, a). \tag{2.1}$$

Here $B(x, a)$ *denotes a ball of radius* a *centering at* x *in* R^l.

Proof. Using Taylor's formula, for any $y, w \in R^l$,

$$\varphi_{x_0x_1}(y+w) = \varphi_{x_0x_1}(y) + T_y\varphi_{x_0x_1}(w) + r_{x_0x_1}(w, y). \tag{2.2}$$

Since E compact, we can find some constant $C > 0$ and $c > 0$, such that for all $y \in E$ and $w \in R^l$ with $\|w\| \leq c$, we have $\|r_{x_0x_1}(w, y)\| < C\|w\|^{1+\kappa}$. We will set also $b = \min_{x\in E,\ i,\ j}\{\alpha_{l,ij}(x)\}$, where $\alpha_{l,ij}$ is the square root of the least eigenvalue of $T_x\varphi_{ij}T_x\varphi_{ij}^*$.

Fix any small $\varepsilon > 0$. Since E is compact and all construction diffeomorphisms satisfy the κ-pinching condition, without loss of generality we can assume ε to be so small that for all pairs (i, j),

$$\|(1+\varepsilon)T_x\varphi_{ij}\| < 1 \quad \text{for all } x \in E, \tag{2.3}$$

$$(1+\varepsilon)^\kappa \alpha_{1,ij}(x)^{1+\kappa} < \alpha_{l,ij}(x) \quad \text{for all } x \in E. \tag{2.4}$$

Pick $\delta > 0$, with $\delta < \min\{c, (b\varepsilon/C)^{1/\kappa}\}$. Thus $\delta^\kappa < \varepsilon\alpha_{l,ij}(x)/C$ for all $x \in E$ and all pairs of (i, j). Let $a \leq \delta$, and pick any $w \in R^l$ with $\|w\| < a$. For any x in E, $\|r_{x_0x_1}(x, w)\| < C\|w\|^{1+\kappa} < Ca^{1+\kappa} \leq aC\delta^\kappa \leq a\varepsilon\alpha_{l,x_0x_1}(x)$ and thus $r_{x_0x_1}(w, x) \in \varepsilon\alpha_{l,x_0x_1}(x)B(0, a)$. Since

$$\varepsilon\alpha_{l,x_0x_1}(x)B(0, a) \subset \varepsilon T_x\varphi_{x_0x_1}B(0, a),$$

it follows from (2.2) that

$$\begin{aligned}\varphi_{x_0x_1}(x+w) &= \varphi_{x_0x_1}(x) + T_x\varphi_{x_0x_1}(w) + r_{x_0x_1}(w, x)\\ &\in \varphi_{x_0x_1}(x) + T_x\varphi_{x_0x_1}B(0, a) + \varepsilon T_x\varphi_{x_0x_1}B(0, a)\\ &= \varphi_{x_0x_1}(x) + (1+\varepsilon)T_x\varphi_{x_0x_1}B(0, a).\end{aligned}$$

This gives (2.1) for $n = 1$. Now the induction hypothesis gives

$$\begin{aligned}\varphi_{x_0x_1\cdots x_n}B(x\,,\,a) &= \varphi_{x_0x_1}\varphi_{x_1\cdots x_n}B(x\,,\,a)\\ &\subset \varphi_{x_0x_1}[\varphi_{x_1\cdots x_n}(x) + (1+\varepsilon)^{n-1}T_x\varphi_{x_1\cdots x_n}B(0\,,\,a)].\end{aligned}$$

Using (2.2),

$$\begin{aligned}&\varphi_{x_0x_1}[\varphi_{x_1\cdots x_n}(x) + (1+\varepsilon)^{n-1}T\varphi_{x_1\cdots x_n}(w)]\\ &\quad = \varphi_{x_0\cdots x_n}(x) + (1+\varepsilon)^{n-1}T\varphi_{x_0\cdots x_n}(w)\\ &\qquad + r_{x_0x_1}((1+\varepsilon)^{n-1}T\varphi_{x_1\cdots x_n}(w)\,,\,\varphi_{x_1\cdots x_n}(x)).\end{aligned}$$

Because of (2.3), $\|(1+\varepsilon)^{n-1}T\varphi_{x_1\cdots x_n}(w)\| < \|w\| < a$, where $w \in B(0\,,\,a)$. Using (2.4), we have

$$\begin{aligned}&\|r_{x_0x_1}((1+\varepsilon)^{n-1}T\varphi_{x_1\cdots x_n}(w)\,,\,\varphi_{x_1\cdots x_n}(x))\|\\ &\quad < C\|(1+\varepsilon)^{n-1}T\varphi_{x_1\cdots x_n}(w)\|^{1+\kappa}\\ &\quad \le C(1+\varepsilon)^{(n-1)(1+\kappa)}\cdot[\alpha_{1\,,\,x_1x_2}(\varphi_{x_2\cdots x_n}(x))\cdots\alpha_{1\,,\,x_{n-1}x_n}(x)\|w\|]^{1+\kappa}\\ &\quad < C(1+\varepsilon)^{n-1}\alpha_{l\,,\,x_1x_2}(\varphi_{x_2\cdots x_n}(x))\cdots\alpha_{l\,,\,x_{n-1}x_n}(x)\|w\|^{1+\kappa}\\ &\quad < (1+\varepsilon)^{n-1}a^{\kappa}C\alpha_{l\,,\,x_1x_2}(\varphi_{x_2\cdots x_n}(x))\cdots\alpha_{l\,,\,x_{n-1}x_n}(x)\|w\|\\ &\quad < \varepsilon(1+\varepsilon)^{n-1}\alpha_{l\,,\,x_0x_1}(\varphi_{x_1\cdots x_n}(x))\cdot\alpha_{l\,,\,x_1x_2}(\varphi_{x_2\cdots x_n}(x))\cdots\alpha_{l\,,\,x_{n-1}x_n}(x)a.\end{aligned}$$

On the other hand $T_x\varphi_{ij}B(0\,,\,a) \supset \alpha_{l\,,\,ij}(x)B(0\,,\,a)$, and it follows that

$$Tx\varphi_{x_0\cdots x_n}B(0\,,\,a) \supset \alpha_{l\,,\,x_0x_1}(\varphi_{x_1\cdots x_n}(x))\cdots\alpha_{l\,,\,x_{n-1}x_n}(x)B(0\,,\,a).$$

Hence $r_{x_0x_1}((1+\varepsilon)^{n-1}T\varphi_{x_1\cdots x_n}(w)\,,\,\varphi_{x_1\cdots x_n}(x)) \in \varepsilon(1+\varepsilon)^{n-1}T_x\varphi_{x_0\cdots x_n}B(0\,,\,a)$. Therefore,

$$\begin{aligned}&\varphi_{x_0x_1}[\varphi_{x_1\cdots x_n}(x) + (1+\varepsilon)^{n-1}T\varphi_{x_1\cdots x_n}(w)]\\ &\quad \in \varphi_{x_0\cdots x_n}(x) + (1+\varepsilon)^{n-1}T_x\varphi_{x_0\cdots x_n}B(0\,,\,a) + \varepsilon(1+\varepsilon)^{n-1}T_x\varphi_{x_0\cdots x_n}B(0\,,\,a)\\ &\quad \subset \varphi_{x_0\cdots x_n}(x) + (1+\varepsilon)^{n}T_x\varphi_{x_0\cdots x_n}B(0\,,\,a).\end{aligned}$$

Thus (2.1) is true for n. This completes the induction process. □

I have learned that Jiang [J] has a distorsion lemma for a regular nonconformal semigroup, which is a semigroup of pinched contracting diffeomorphisms. His version is stronger than our version here. However, for our purpose of estimating Hausdorff dimensions, our version is strong enough.

Proposition 2.2. *If all construction diffeomorphisms φ_{ij} are $C^{1+\kappa}$ and satisfy the κ-pinching condition where $0 < \kappa \le 1$, and if the topological pressure $P(\sigma\,,\,\lambda_t) < 0$ where σ is the shift map in Σ, then the Hausdorff dimension* $\mathrm{HD}(E) \le t$.

Proof. Choose small $\varepsilon > 0$ with $P(\sigma\,,\,\lambda_t) < -2t\varepsilon$, satisfying both (2.3) and (2.4). By Lemma 2.1, there exists $\delta > 0$ such that (2.1) holds for all integer $n > 0$ and each $x \in E$, when $0 < a < \delta$.

We fix $a < \delta$ small enough, and a positive integer n big enough, such that (see [W] for notation) $\log P_n(\sigma\,,\,\lambda_t\,,\,a) < -2nt\varepsilon$. Recall that π is Hölder continuous. Suppose that γ is the exponent such that there exists a constant D

with $|\pi(\mathbf{x})-\pi(\mathbf{y})| < D\cdot d(\mathbf{x}, \mathbf{y})^\gamma$ for all $\mathbf{x}$, $\mathbf{y}$ in Σ. Fix $a' < \min\{D^{-1/\gamma}a^{1/\gamma}, a\}$. Pick m with $2^{-m-1} < a' \le 2^{-m}$. Let

$$K' = \{(x_0, \dots, x_{m+n})| \text{ there exists } \mathbf{x} \in \Sigma \text{ with } \mathbf{x} = (x_0, \dots, x_{m+n}, \dots)\}.$$

Choose for each word $(x_0, \dots, x_{m+n})$ in K' a point $\mathbf{x}$ in Σ with the initial of $x_0, \dots, x_{m+n}$ to form a subset K of Σ. The subset K is (n, a') separated and is maximal in the sense that one cannot add another point to K such that it is still (n, a') separated. Thus, the collection $\{\sigma^{-n}B(\sigma^n\mathbf{x}, a') \,|\, \mathbf{x} \in K\}$ is an open cover for Σ. Notice that $\pi\mathbf{x} = \varphi_{x_0x_1}\pi\sigma\mathbf{x}$. Since $\pi B(\mathbf{x}, a') \subset B(\pi(\mathbf{x}), a)$ and $\pi\{\sigma^{-n}B(\sigma^n\mathbf{x}, a') \,|\, \mathbf{x} \in K\} \subset \{\varphi_{x_0x_1\cdots x_n}B(\pi\sigma^n\mathbf{x}, a) \,|\, \mathbf{x} = (x_0, x_1, \dots, x_n \dots) \in K\}$ follows, $\{\varphi_{x_0x_1\cdots x_n}B(\pi\sigma^n\mathbf{x}, a) \,|\, \mathbf{x} = (x_0, x_1, \dots, x_n \dots) \in K\}$ is an open cover for $E = \bigcup_{i=1}^l E_i$.

Using (2.1) of Lemma 2.1,

$$\varphi_{x_0\cdots x_n}B(\pi\sigma^n\mathbf{x}, a) \subset \varphi_{x_0\cdots x_n}(\pi\sigma^n\mathbf{x}) + (1+\varepsilon)^n T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n}B(0, a). \tag{2.5}$$

The right side of (2.5) is an ellipsoid with axes $\{a(1+\varepsilon)^n\alpha_j(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n}) \,|\, 1 \le j \le l\}$. Pick j with $j-1 \le t < j$. Then that ellipsoid can be covered by

$$\begin{aligned} &C\cdot\alpha_1(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n})\cdots\alpha_j(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n})/\alpha_j(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n})^j \\ &\quad = C\cdot\omega_{j-1}(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n})\alpha_j^{-j+1}(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n}) \end{aligned}$$

balls of radius $a(1+\varepsilon)^n\alpha_j(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n})$, where the constant $C > 0$ depends only on the dimension of R^l. Now we calculate the Hausdorff t-measure of E, using the smaller balls of radius $a(1+\varepsilon)^n\cdot\alpha_j(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n}) < a$ to cover the open set $\varphi_{x_0\cdots x_n}B(\pi\sigma^n\mathbf{x}, a)$. If $\{P_i : i \in I\}$ is an open cover for E where P_i is a ball of radius r_i, then we define $|I| = \max_{i\in I} r_i$ and $\mu(a, t) = \inf_{|I|<a}\sum_{i\in I} r_i^t$. We have

$$\begin{aligned} \mu(a, t) &\le \sum_{\mathbf{x}\in K} C\omega_{j-1}(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n})\alpha_j^{-j+1} \\ &\qquad\qquad \times (T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n})[a(1+\varepsilon)^n\alpha_j(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n})]^t \\ &= (1+\varepsilon)^{nt}a^tC\sum_{\mathbf{x}\in K}\omega_t(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_0\cdots x_n}) \\ &\le (1+\varepsilon)^{nt}C\sum_{\mathbf{x}\in K}\omega_t(T_{\pi\sigma\mathbf{x}}\varphi_{x_0x_1})\omega_t(T_{\pi\sigma^2\mathbf{x}}\varphi_{x_1x_2})\cdots\omega_t(T_{\pi\sigma^n\mathbf{x}}\varphi_{x_{n-1}x_n}) \\ &= (1+\varepsilon)^{nt}C\sum_{\mathbf{x}\in K}\exp[\lambda_t(\mathbf{x}) + \lambda_t(\sigma\mathbf{x}) + \cdots + \lambda_t(\sigma^{n-1}\mathbf{x})] \\ &\le (1+\varepsilon)^{nt}CP_n(\sigma, \lambda_t, a') \le (1+\varepsilon)^{nt}CP_n(\sigma, \lambda_t, a) \\ &\le C\exp(nt\varepsilon)\exp(-2nt\varepsilon) \to 0, \end{aligned}$$

as $n \to \infty$. Thus $\mu(a, t) = 0$. Since a can be arbitrarily small, $\mu(t) = 0$. It follows that $\mathrm{HD}(E) \le t$. $\square$

Proof of Theorem 1. $P(\sigma, \lambda_t)$ is a decreasing function of t's since E is compact and λ_t is strictly decreasing with respect to t. So there is only one real number t such that $P(\sigma, \lambda_t) = 0$. Also, the unique t with $P(\sigma, \lambda_t) = 0$ is equal to $\inf\{t : P(\sigma, \lambda_t) < 0\}$. Consequently, we have $\mathrm{HD}(E) \le t$ where $P(\sigma, \lambda_t) = 0$.

3. The lower bound

Proof of Theorem 2. Notice that for each t, the map λ_t is Hölder continuous on Σ. So there exists an equilibrium state μ for λ_t, in the sense that

$$P(\sigma, \lambda_t) = h_\mu(\sigma) + \int \lambda_t \, d\mu.$$

Fix any $\rho > 0$, let us estimate the μ-measure of a ball $B(z, \rho)$ centered at z with radius ρ. For each $\mathbf{x} \in \Sigma$ choose the unique $n = n(\mathbf{x}) \geq 0$ such that the diameters satisfy

$$\operatorname{diam}(U_n(\mathbf{x})) \leq \rho < \operatorname{diam}(U_{n-1}(\mathbf{x})).$$

Lemma 3.1. *There exists a constant $c > 0$ such that for all $\mathbf{x} \in \Sigma$, the open set $U_{n(\mathbf{x})}(\mathbf{x})$ is contained in a ball of radius ρ and contains a ball of radius $c\rho^{1+\kappa}$.*

Proof. It is clear that $U_n(\mathbf{x})$ is contained in a ball of radius ρ. Since the radius of $U_n(\mathbf{x})$ decreases to 0 as n grows to infinity, without loss of generality we can assume the maximum diameter R of all U_i is less than the number δ given in Lemma 2.1. Also pick r small enough that each U_i contains a ball of radius r. Then $U_n(\mathbf{x})$ contains a ball of radius

$$r \cdot \alpha_{l, x_0 x_1}(\pi\sigma\mathbf{x}) \cdots \alpha_{l, x_{n-1} x_n}(\pi\sigma^n\mathbf{x}) > r \cdot \alpha^{1+\kappa}_{1, x_0 x_1}(\pi\sigma\mathbf{x}) \cdots \alpha^{1+\kappa}_{1, x_{n-1} x_n}(\pi\sigma^n\mathbf{x}).$$

But on the other hand

$$\rho \leq \operatorname{diam}(U_{n-1}(\mathbf{x})) \leq \alpha_{1, x_0 x_1}(\pi\sigma\mathbf{x}) \cdots \alpha_{1, x_{n-2} x_{n-1}}(\pi\sigma^{n-1}\mathbf{x}) R,$$

which implies that $\alpha_{1, x_0 x_1}(\pi\sigma\mathbf{x}) \cdots \alpha_{1, x_{n-1} x_n}(\pi\sigma^n\mathbf{x}) \geq \alpha_1 \rho / R$ where the constant $\alpha_1 = \min_{y \in E, i, j}\{\alpha_{1, ij}(y)\} > 0$ does not depend on either n or $\mathbf{x}$. Therefore $U_n(\mathbf{x})$ contains a ball of radius $> r\rho^{1+\kappa}\alpha_1^{1+\kappa}/R^{1+\kappa}$. Writing $c = \alpha_1^{1+\kappa} r / R^{1+\kappa}$ a constant, $U_n(\mathbf{x})$ contains a ball of radius $c\rho^{1+\kappa}$ as desired. □

For two points $\mathbf{x}, \mathbf{y} \in \Sigma$, since the construction maps satisfy the open set condition, $U_{n(\mathbf{x})}(\mathbf{x})$ and $U_{n(\mathbf{y})}(\mathbf{y})$ are either equal or disjoint. Let $\Gamma \subset \Sigma$ be a subset such that $\{U_{n(\mathbf{x})}(\mathbf{x}) \mid \mathbf{x} \in \Gamma\}$ is a disjoint collection which contains all $U_{n(\mathbf{x})}(\mathbf{x})$ for $\mathbf{x} \in \Sigma$. Notice that $\{\overline{U}_{n(\mathbf{x})}(\mathbf{x}) \mid \mathbf{x} \in \Gamma\}$ covers E.

Lemma 3.2 (similar to [H, 5.3(a)]). *At most $3^l c^{-l} \rho^{-\kappa l}$ of $\{\overline{U}_{n(\mathbf{x})}(\mathbf{x}) \mid \mathbf{x} \in \Gamma\}$ can meet $B(z, \rho)$.*

Proof. Suppose that $\overline{V}_1, \ldots, \overline{V}_m$ in $\{\overline{U}_{n(\mathbf{x})}(\mathbf{x}) \mid \mathbf{x} \in \Gamma\}$ meet $B(z, \rho)$. Then each of them is a subset of $B(z, 3\rho)$. By the definition of Γ the sets in the collection $\{\overline{U}_{n(\mathbf{x})}(\mathbf{x}) \mid \mathbf{x} \in \Gamma\}$ are disjoint. Comparing the volumes we have $mJc^l\rho^{l(1+\kappa)} \leq J3^l\rho^l$ where J is the volume of a unit ball in R^l. Hence $m \leq 3^l c^{-l}\rho^{-\kappa l}$. □

Let $C_n(\mathbf{x}) = \{\mathbf{y} = (y_0, y_1, \ldots) \in \Sigma \mid y_0 = x_0, \ldots, y_n = x_n\}$ be the n cylinder. Recall that μ is a Gibbs measure (see [Bo] for a discussion or [B2] for a summary). There exists a constant $d > 0$, with

$$\mu(C_n(\mathbf{x})) \in [d^{-1}, d] \cdot \exp(-P(\sigma, \lambda_t)n + S_n\lambda_t(\mathbf{x})), \quad \text{for each cylinder } C_n(\mathbf{x}) \text{ in } \Sigma.$$

Thus $\mu(C_n(\mathbf{x})) \in [d^{-1}, d] \cdot \exp(S_n\lambda_t(\mathbf{x}))$, since $P(\sigma, \lambda_t) = 0$. So,

$$\begin{aligned}\mu(C_n(\mathbf{x})) &\le d \exp S_n\lambda_t(\mathbf{x}) \\ &\le d[\alpha_{1, x_0x_1}(\pi\sigma\mathbf{x}) \cdots \alpha_{1, x_{n-1}x_n}(\pi\sigma^n\mathbf{x})]^t \\ &\le d[\alpha_{l, x_0x_1}(\pi\sigma\mathbf{x}) \cdots \alpha_{l, x_{n-1}x_n}(\pi\sigma^n\mathbf{x})]^{t/(1+\kappa)} \\ &\le d \cdot [\operatorname{diam}(U_{n(\mathbf{x})}/r]^{t/(1+\kappa)}.\end{aligned}$$

Hence if $n = n(\mathbf{x})$ we obtain

$$\mu(C_{n(\mathbf{x})}(\mathbf{x})) \le \frac{d\rho^{t/(1+\kappa)}}{r^{t/(1+\kappa)}}.$$

Noticing $\pi C_n(\mathbf{x}) \supset \overline{U}_n(\mathbf{x}) \cap E$, by Lemma 3.2,

$$\pi_*\mu(B(z, \rho)) \le [3^l c^{-l} d r^{-t/(1+\kappa)}]\rho^{t/(1+\kappa)-l\kappa}.$$

By the Frostman lemma (see [K] for a proof), $\mathrm{HD}(E) \ge t/(1+\kappa) - l\kappa$.

4. Some continuity of the Hausdorff dimension in C^1 topology

The construction of the self-similar set E_φ depends on the contracting diffeomorphisms $\{\varphi_{ij}\}$. Now let us fix $0 < \beta \le 1$, and consider a C^1 perturbation to a $C^{1+\beta}$ conformal construction with diffeomorphisms $\{\varphi_{ij}\}$, and obtain another matrix of contracting diffeomorphisms $\{\psi_{ij}\}$, which is not necessarily conformal. Denote the new self-similar set for ψ by E_ψ. Define $d_{C^1}(\varphi, \psi) = \max_{i,j} d_{C^1}\{(\varphi_{ij}, \psi_{ij})\}$, where the latter d_{C^1} is the C^1 metric. Note that for any $\kappa < \beta$, when ψ is sufficiently C^1 close to φ, ψ must be $C^{1+\kappa}$ and also κ-pinched. The following theorem is a corollary of Theorems 1 and 2, which states that at a $C^{1+\beta}$ conformal construction satisfying the open set condition for self-similar sets, the Hausdorff dimension $\mathrm{HD}(E_\psi)$ depends continuously on $\{\psi_{ij}\}$ in C^1 topology.

Theorem 4.1. *Let $\{\varphi_{ij}\}$ be a matrix of $C^{1+\beta}$ conformal construction diffeomorphisms for the self-similar set E_φ, satisfying the open set condition. For any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $C^{1+\beta}$ construction ψ satisfying the open set condition, with $d_{C^1}(\varphi, \psi) < \delta$, we have $|\mathrm{HD}(E_\varphi) - \mathrm{HD}(E_\psi)| < \varepsilon$.*

Proof. Let $\lambda_{\varphi,s}(\mathbf{x}) = \log\omega_s(T\varphi_{x_0x_1}(\pi\sigma\mathbf{x}))$ and $\lambda_{\psi,s}(\mathbf{x}) = \log\omega_s(T\psi_{x_0x_1}(\pi\sigma\mathbf{x}))$ be two real functions on Σ as defined in Definition 1.1. Let t be such that $P(\sigma, \lambda_{\varphi,t}) = 0$. Because φ_{ij}'s are conformal, the Hausdorff dimension of E_φ equals t. Also, remark that $P(\sigma, \lambda_{\varphi,t+\varepsilon}) < 0$ for any $\varepsilon > 0$.

Now fix any $\varepsilon > 0$. Let $\kappa = \min\{\beta, \varepsilon/4l\}$ and let

$$\varepsilon' = \tfrac{1}{2}\min\{-P(\sigma, \lambda_{\varphi,t+\varepsilon}), P(\sigma, \lambda_{\varphi,t-\varepsilon/4})\} > 0. \tag{4.1}$$

Since φ is $C^{1+\beta}$ and conformal, there is $\delta > 0$ such that a $C^{1+\beta}$ diffeomorphism ψ is $C^{1+\kappa}$ and κ-pinched with $|\lambda_{\varphi,s}(\mathbf{x}) - \lambda_{\psi,s}(\mathbf{x})| < \varepsilon'$ for all $s \in [0, l]$, if $d_{C^1}(\varphi, \psi) < \delta$.

Then $P(\sigma, \lambda_{\psi,t+\varepsilon}) < P(\sigma, \lambda_{\varphi,t+\varepsilon} + \varepsilon') \le P(\sigma, \lambda_{\varphi,t+\varepsilon}) + \varepsilon' < 0$. So $\mathrm{HD}(E_\psi) \le t + \varepsilon = \mathrm{HD}(E_\varphi) + \varepsilon$, by Proposition 2.2.

On the other hand, by (4.1), when $d_{C^1}(\varphi, \psi) < \delta$, we have

$$P(\sigma, \lambda_{\psi,t-\varepsilon/4}) > P(\sigma, \lambda_{\varphi,t-\varepsilon/4} - \varepsilon') \ge P(\sigma, \lambda_{\varphi,t-\varepsilon/4}) - \varepsilon' > 0.$$

So we have some $s > t - \varepsilon/4$ with $P(\sigma, \lambda_{\psi,s}) = 0$ since $P(\sigma, \lambda_{\psi,s})$ is strictly decreasing with respect to s. By Theorem 2, $\mathrm{HD}(E_\psi) \geq s/(1+\kappa) - l\kappa \geq s/(1+\varepsilon/4l) - l(\varepsilon/4l) > s(1-\varepsilon/4l) - \varepsilon/4 > s - \varepsilon/2 > t - \varepsilon$. It then follows that $\mathrm{HD}(E_\psi) \geq t - \varepsilon = \mathrm{HD}(E_\varphi) - \varepsilon$. □

We say a construction φ with diffeomorphisms $\{\varphi_{ij}\}$ satisfies the strong open set condition if there are open sets $U_1, \dots, U_l$ in R^l with $\varphi_{ij}(\overline{U}_j) \subset U_i$ for all i, j. If the construction φ satisfies the strong open set condition, then ψ must also satisfy the strong open set condition if it is C^1 close enough to φ. Thus we have obtained an immediate corollary of the above Theorem 4.1:

Corollary 4.2. *Let $\{\varphi_{ij}\}$ be a matrix of $C^{1+\beta}$ conformal construction diffeomorphisms for the self-similar set E_φ, satisfying the strong open set condition. For any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $C^{1+\beta}$ construction ψ with $d_{C^1}(\varphi, \psi) < \delta$, we have $|\mathrm{HD}(E_\varphi) - \mathrm{HD}(E_\psi)| < \varepsilon$.*

Finally we have a remark on the continuity of the Hausdorff dimension in C^1 topology at nonconformal constructions.

Remark 4.3. The following example shows that if the "conformal" condition for the construction diffeomorphisms $\{\varphi_{ij}\}$ fails, then the results in Theorem 4.1 and Corollary 4.2 can be false. The example is derived from Example 9.10 of Falconer [F, pp. 127–128].

Let $S, T_\lambda \colon R^2 \to R^2$ be given by

$$S(x, y) = (x/2,\ y/3 + 2/3), \qquad T_\lambda(x, y) = (x/2 + \lambda,\ y/3)$$

where $\lambda \in [0, 1/2)$ and $(x, y) \in R^2$. Let $\varphi_{11} = \varphi_{21} = S$, $\varphi_{12} = \varphi_{22} = T_0$. Take $\psi_\lambda = \{\psi_{ij,\lambda}\}$ where $\psi_{11,\lambda} = \psi_{21,\lambda} = S$ and $\psi_{12,\lambda} = \psi_{22,\lambda} = T_\lambda$. The strong open set condition is met for $\{\varphi_{ij}\}$. In fact, if we let $U_1 = U_2 = (-1/8, 9/8)^2 \subset R^2$ then $\varphi_{ij}(\overline{U}_j) \subset U_i$.

Let E_φ, E_{ψ_λ} be the self-similar sets for φ and ψ_λ. Considering the projection of E_{ψ_λ} to the x-axis, one knows that $\mathrm{HD}(E_{\psi_\lambda}) \geq 1$ for $\lambda > 0$. But E_φ is a Cantor set contained in the y-axis with the Hausdorff dimension $\mathrm{HD}(E_\varphi) = (\log 2)/\log 3 < 1$. Since $d_{C^1}(\psi_\lambda, \varphi) = \lambda$, letting $\lambda \to 0$ we know the Hausdorff dimension is not continuous at φ. We notice that $\{\varphi_{ij}\}$ are not conformal although $\{\varphi_{ij}\}$ and $\{\psi_{ij,\lambda}\}$ are all $3/4$ pinched.

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Department of Mathematics, University of Florida, Gainesville, Florida 32611
E-mail address: xgu@math.ufl.edu