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ON STRONGLY DISCRETE SUBSETS OF ω^*

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ABSTRACT. We prove that it is consistent with Martin's Axiom and $\neg$CH that there is a strongly discrete subspace $A \subseteq \omega^*$ of cardinality $\aleph_1$ such that the closure of A is not homeomorphic with $\beta\omega_1$. We also prove that MA and $\neg$CH imply that there is no convergent strongly discrete subset of ω^*.

1. INTRODUCTION

We consider closures of strongly discrete subsets of ω^*, the remainder of Čech-Stone compactification of ω. A set $D \subseteq \omega^*$ is strongly discrete if for every $p \in D$ there is an open neighbourhood $p \in U_p$ such that $U_p \cap U_q = \varnothing$ for $p \neq q$.

It is well known that the closure of any countable, discrete subspace of ω^* is homeomorphic with $\beta\omega$. Moreover, the cardinality assumption is essential. By a result of [2] there exists a discrete subspace $D \subseteq \omega^*$ of cardinality $\aleph_1$ which is convergent, i.e., there is $p \in \omega^*$ such that every neighbourhood of p contains all but countably many elements of D. The closure of such a D is not homeomorphic with $\beta\omega_1$.

However, it is consistent with Martin's Axiom (MA) and, in fact, follows from the Proper Forcing Axiom that the closure of every strongly discrete subspace of ω^* of cardinality $\aleph_1$ is homeomorphic with $\beta\omega_1$ (see [6, 5, 3]). In this paper we show that this result does not follow from MA itself. To prove this one might try to show the consistency of MA with 'There exists a strongly discrete, convergent subspace of ω^* of cardinality $\aleph_1$'. We prove below that this is not possible because MA implies that there are no such subsets.

Throughout the paper we use Boolean-algebraic terminology. We consider ω^* to be the space of nonprincipal ultrafilters on ω with the Stone topology. In particular, $D = \{p_\alpha : \alpha \in \omega_1\}$ is strongly discrete if there exists an almost disjoint family $\{a_\alpha : \alpha \in \omega_1\} \subseteq \mathscr{P}(\omega)$ such that $a_\alpha \in p_\alpha$ for every $\alpha \in \omega_1$. The set-theoretical translation of the problem we consider is given by the following theorem (see [4]): *The closure of a strongly discrete subspace $D = \{p_\alpha : \alpha \in \omega_1\}$ is homeomorphic with $\beta\omega_1$ if for every $E \subseteq \omega_1$ there is $c \subseteq \omega$ such that*

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$\alpha \in E$ *if and only if* $c \in p_\alpha$. We say that c separates $\{p_\alpha : \alpha \in E\}$ and $\{p_\alpha : \alpha \in (\omega_1 - E)\}$.

Most of our notation is standard. We say that $F \subseteq \mathcal{P}(\omega)$ has the finite intersection property if the intersection of any finitely many elements of F is infinite. If F has the finite intersection property, then we write $\langle F \rangle$ to denote the filter generated by F and the cofinite subsets of ω. For $a \subseteq \omega$ we write $\langle a \rangle$ instead of $\langle \{a\} \rangle$.

Theorem 1 (MA+$\neg$CH). *There is no convergent strongly discrete subspace of ω^* of cardinality $\aleph_1$.*

Proof. Suppose that $A = \{p_\alpha : \alpha \in \omega_1\}$ is strongly discrete and converges to $p \in \omega^*$. Let $\{a_\alpha : \alpha \in \omega_1\}$ be an almost disjoint family such that $a_\alpha \in p_\alpha$ for every $\alpha \in \omega_1$, and let $\omega_1 = E_0 \cup E_1 \cup E_2$ be a partition of ω_1 into three uncountable sets. We define a forcing $\mathbb{Q}$. A condition in $\mathbb{Q}$ is a triple (s_0, s_1, s_2) of finite partial functions such that $\mathrm{dom}(s_i) \subseteq E_i$ and $\mathrm{ran}(s_i) \subseteq \omega$ for all i and such that the intersection $S_0 \cap S_1 \cap S_2$ is empty, where $S_i = \bigcup\{a_\xi - s_i(\xi) : \xi \in \mathrm{dom}(s_i)\}$. A condition $t = (t_0, t_1, t_2)$ is an extension of (s_0, s_1, s_2) if $t_i \subseteq s_i$ for $i < 3$.

Note that for every $i < 3$ and $\alpha \in E_i$ the set $D^i_\alpha = \{(s_0, s_1, s_2) : \alpha \in \mathrm{dom}(s_i)\}$ is dense in $\mathbb{Q}$. Let G be a generic set with respect to $\{D^i_\alpha : i < 3 \text{ and } \alpha \in E_i\}$. For $i < 3$ define

$$C_i = \bigcup\{a_\alpha - s_i(\alpha) : (s_0, s_1, s_2) \in G \text{ and } \alpha \in E_i\}.$$

Note that $C_i \in p_\alpha$ for every $\alpha \in E_i$. Since each E_i is uncountable and A converges to p, we have $C_0, C_1, C_2 \in p$. On the other hand $C_0 \cap C_1 \cap C_2 = \varnothing$, a contradiction.

To finish the proof it is enough to show that $\mathbb{Q}$ is c.c.c. Let $\{(s^\alpha_0, s^\alpha_1, s^\alpha_2) : \alpha \in \omega_1\}$ be an uncountable subset of $\mathbb{Q}$. By thinning out we can assume that for every $i < 3$ the collection $\{\mathrm{dom}(s^\alpha_i) : \alpha \in \omega_1\}$ forms a Δ-system with root Δ_i and $s^\alpha_i \restriction_{\Delta_i} = s^\beta_i \restriction_{\Delta_i}$ for $\alpha, \beta \in \omega_1$. Let $I_\alpha = (S^\alpha_0 \cap S^\alpha_1,\ S^\alpha_0 \cap S^\alpha_2,\ S^\alpha_1 \cap S^\alpha_2)$. Since I_α is a triple of finite subsets of ω, we can assume that $I_\alpha = I_\beta$ for $\alpha, \beta \in \omega_1$. Now we show that any two conditions are compatible. Let α, $\beta \in \omega_1$, and let $t = (t_0, t_1, t_2)$ be defined by $t_i = s^\alpha_i \cup s^\beta_i$ for $i < 3$. Note that

$$T_0 \cap T_1 \cap T_1 \subseteq (S^\alpha_0 \cap S^\alpha_1 \cap S^\alpha_2) \cup (S^\beta_0 \cap S^\beta_1 \cap S^\beta_2).$$

Hence $(t_0, t_1, t_2) \in \mathbb{Q}$ and it is a common extension of $(s^\alpha_0, s^\alpha_1, s^\alpha_2)$ and $(s^\beta_0, s^\beta_1, s^\beta_2)$. This proves that $\mathbb{Q}$ is c.c.c. $\square$

2. The main result

Let $\mathcal{F} = \{F_\alpha : \alpha \in \omega_1\}$ and $\mathcal{G} = \{G_\alpha : \alpha \in \omega_1\}$ be families of filters on ω. The following is the key property we will deal with.

Definition. We say that $\langle \mathcal{F}, \mathcal{G} \rangle$ is good if for every $c \subseteq \omega$ the set $A_c = \{\xi \in \omega_1 : c \in F_\xi \text{ and } (\omega - c) \in G_\xi\}$ is countable.

If F_α and G_α have the finite intersection property, then we say that $\langle \mathcal{F}, \mathcal{G} \rangle$ is good if the families of filters $\{\langle F_\alpha \rangle : \alpha \in \omega_1\}$ and $\{\langle G_\alpha \rangle : \alpha \in \omega_1\}$ form a good pair. In particular if $A = \{a_\alpha : \alpha \in \omega_1\}$ and $B = \{b_\alpha : \alpha \in \omega_1\}$ are such that $A \cup B$ is an almost disjoint family in $\mathscr{P}(\omega)$, then $\langle A, B\rangle$ is good if for every $c \subseteq \omega$ the set $A_c = \{\xi \in \omega_1 : a_\xi \subseteq^* c \text{ and } b_\xi \cap c =^* \varnothing\}$ is countable. Let $\overline{\mathscr{F}} = \{\overline{F}_\alpha : \alpha \in \omega_1\}$ and $\overline{\mathscr{G}} = \{\overline{G}_\alpha : \alpha \in \omega_1\}$. We say that $\langle \overline{\mathscr{F}}, \overline{\mathscr{G}}\rangle$ extends $\langle \mathscr{F}, \mathscr{G}\rangle$ if $F_\alpha \subseteq \overline{F}_\alpha$ and $G_\alpha \subseteq \overline{G}_\alpha$ for $\alpha \in \omega_1$. The goal of this section is to prove the following

Theorem 2. *It is consistent with $MA + \neg CH$ that there is a strongly discrete subset of ω^* of cardinality $\aleph_1$ whose closure is not homeomorphic with $\beta\omega_1$.*

Proof. As pointed out in the introduction it is enough to construct a model of MA with subsets $\mathscr{F} = \{p_\alpha : \alpha \in \omega_1\}$ and $\mathscr{G} = \{q_\alpha : \alpha \in \omega_1\}$ of ω^* such that $\mathscr{F} \cup \mathscr{G}$ is a strongly discrete subspace and there is no $c \subseteq \omega$ which separates $\mathscr{F}$ and $\mathscr{G}$.

We start with a model V satisfying $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$ and define a finite support iteration $\langle \mathbb{P}_\alpha : \alpha \in \omega_2\rangle$ of c.c.c. posets in order to get Martin's Axiom, i.e., by suitable bookkeeping we make sure that every potential c.c.c. poset is considered at some intermediate stage. We iterate only posets of cardinality $\aleph_1$, thus intermediate models, $V_\alpha = V^{\mathbb{P}_\alpha}$, also satisfy $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$.

The idea of the proof is as follows. First, we construct $A = \{a_\xi : \xi \in \omega_1\}$ and $B = \{b_\xi : \xi \in \omega_1\}$ such that $A \cup B$ is an almost disjoint family and $\langle A, B\rangle$ is good. Next we find a pair $\langle \mathscr{F}^0, \mathscr{G}^0\rangle$ of families of ultrafilters which is good and extends $\langle A, B\rangle$. Then we proceed by induction. Along with the iteration we define $\langle \mathscr{F}^\alpha, \mathscr{G}^\alpha\rangle$ to be a $\mathbb{P}_\alpha$-name for a pair of families of filters $\mathscr{F}^\alpha = \{p^\alpha_\xi : \xi \in \omega_1\}$ and $\mathscr{G}^\alpha = \{q^\alpha_\xi : \xi \in \omega_1\}$ such that for all $\alpha \in \omega_2$ the following hold.

(1) If $\beta < \alpha$, then $\langle \mathscr{F}^\alpha, \mathscr{G}^\alpha\rangle$ extends $\langle \mathscr{F}^\beta, \mathscr{G}^\beta\rangle$.
(2) $\langle \mathscr{F}^\alpha, \mathscr{G}^\alpha\rangle$ is good.
(3) If $\alpha \in \omega_2$ is odd, then $p^\alpha_\xi = \langle x_\xi\rangle$ and $q^\alpha_\xi = \langle y_\xi\rangle$, where $x_\xi \subseteq^* f$ for all $f \in p^{\alpha-1}_\xi$ and $y_\xi \subseteq^* g$ for all $g \in q^{\alpha-1}_\xi$.
(4) If α is even, then $\mathscr{F}^\alpha$ and $\mathscr{G}^\alpha$ are families of ultrafilters.

Note that $\langle \mathscr{F}^{\omega_2}, \mathscr{G}^{\omega_2}\rangle$ is as required in V_{ω_2}. Indeed, $A \cup B$ witnesses that $\mathscr{F}^{\omega_2} \cup \mathscr{G}^{\omega_2}$ is a strongly discrete subspace and, since $\langle \mathscr{F}^{\omega_2}, \mathscr{G}^{\omega_2}\rangle$ is good, no $c \subseteq \omega$ can separate it.

Now let us prove that the construction can be carried out. Using CH in the ground model it is not difficult to construct $A = \{a_\xi : \xi \in \omega_1\}$ and $B = \{b_\xi : \xi \in \omega_1\}$ such that $A \cup B$ is an almost disjoint family and $\langle A, B\rangle$ is good. We use the following lemma to find suitable $\mathscr{F}^0$ and $\mathscr{G}^0$.

Lemma 1 (CH). *Let $\mathscr{F} = \langle F_\xi : \xi \in \omega_1\rangle$ and $\mathscr{G} = \langle G_\xi : \xi \in \omega_1\rangle$ be families of filters such that $\langle \mathscr{F}, \mathscr{G}\rangle$ is good. Then there is an extension $\langle \overline{\mathscr{F}}, \overline{\mathscr{G}}\rangle$ of $\langle \mathscr{F}, \mathscr{G}\rangle$ such that $\langle \overline{\mathscr{F}}, \overline{\mathscr{G}}\rangle$ is good and $\overline{\mathscr{F}}$ and $\overline{\mathscr{G}}$ are families of ultrafilters.*

Suppose now that $\mathbb{P}_\alpha$ and $\langle \mathscr{F}^\alpha, \mathscr{G}^\alpha\rangle$ have been defined for all $\alpha < \gamma$ such that (1)–(4) hold. We have to define $\mathbb{P}_\gamma$ and $\langle \mathscr{F}^\gamma, \mathscr{G}^\gamma\rangle$.

Assume first that $\gamma = \alpha + 1$ is a successor. Working in V_α we define a c.c.c. poset $\mathbb{Q}$ and put $\mathbb{P}_\gamma = \mathbb{P}_\alpha * \mathbb{Q}$. Then in $V_\alpha^{\mathbb{Q}}$ we define $\langle \mathscr{F}^\gamma, \mathscr{G}^\gamma\rangle$ and show that the inductive conditions are satisfied.

Case 1. α is even. We have to make sure that condition (3) is satisfied in V_γ, i.e., we have to diagonalize the ultrafilters appearing in $\mathcal{F}^\alpha$ and $\mathcal{G}^\alpha$. First let us define a well-known forcing and list some of its basic properties. Let U be a set with the finite intersection property. Define $\mathbb{Q}(U) = \{(s, K) : s \subseteq \omega,\ K \subseteq F, \text{ and } s, K \text{ finite}\}$ and order it by $(s, K) \leq (r, L)$ if $r \subseteq s$, $L \subseteq K$, and $s - r \subseteq \bigcap L$. $\mathbb{Q}(U)$ is a standard poset for diagonalizing U, i.e., if $G(U)$ is $\mathbb{Q}(U)$-generic and $x_U = \bigcup\{s : \exists K\ (s, K) \in G(U)\}$, then $x_U \subseteq^* f$ for every $f \in F$. Recall the following facts about $\mathbb{Q}(U)$.

- $\mathbb{Q}(U)$ has the c.c.c.; moreover, if W is another set with the finite intersection property, then $\mathbb{Q}(U) \times \mathbb{Q}(W)$ has the c.c.c. In this case we denote by x_U and y_W generic reals given by $\mathbb{Q}(U)$ and $\mathbb{Q}(W)$ respectively.
- If $e \subseteq \omega$ is such that $U \cup \{e\}$ has the finite intersection property, then the set $x_U \cap e$ is infinite.
- If $\langle \mathbb{R}_\xi : \xi \in \omega_1 \rangle$ is a finite support iteration such that for every $\xi \in \omega_1$ $\mathbb{R}_{\xi+1} = \mathbb{R}_\xi * \mathbb{S}_\xi$, where $\mathbb{S}_\xi$ is (a name for) $\mathbb{Q}(U_\xi) \times \mathbb{Q}(W_\xi)$ for some U_ξ, W_ξ, then $\mathbb{R}_{\omega_1}$ is c.c.c. Moreover, for all $\alpha \leq \omega_1$ $\mathbb{R}_\alpha$ is σ-centered; in particular, it remains c.c.c. in every c.c.c. extension of the universe.

We work in V_α. Define $\mathbb{Q} = \mathbb{R}_{\omega_1}$, where $\langle \mathbb{R}_\xi : \xi \in \omega_1 \rangle$ is as above with $U_\xi = p_\xi^\alpha$ and $W_\xi = q_\xi^\alpha$. Note that $\mathbb{Q}$ is c.c.c. and has size $\aleph_1$ as required. To simplify the notation let $x_\xi = x_{p_\xi^\alpha}$ and $y_\xi = y_{q_\xi^\alpha}$ for $\xi \in \omega_1$. Working in $V_\gamma = V_\alpha^{\mathbb{Q}}$ we define $p_\xi^\gamma = \langle x_\xi \rangle$ and $q_\xi^\gamma = \langle y_\xi \rangle$ for $\xi \in \omega_1$. We claim that $\langle \mathcal{F}^\gamma, \mathcal{G}^\gamma \rangle$ is good in V_γ. This follows from the following results.

Lemma 2 (CH). *Let $F = \{x_\xi : \xi \in \omega_1\}$ and $G = \{y_\xi : \xi \in \omega_1\}$ be such that $F \cup G$ is an almost disjoint family and $\langle F, G \rangle$ is good. Suppose that $\mathbb{R}$ is a c.c.c. forcing such that $\Vdash_{\mathbb{R}}$ '$\langle F, G \rangle$ is not good'. Then there is a c.c.c. forcing $\mathbb{Q}$ such that in $V^{\mathbb{Q}}$ $\mathbb{R}$ is not c.c.c. and $\langle F, G \rangle$ is good.*

Corollary (CH). *Let $\mathcal{F} = \{F_\xi : \xi \in \omega_1\}$ and $\mathcal{G} = \{G_\xi : \xi \in \omega_1\}$ be families of filters such that $\langle \mathcal{F}, \mathcal{G} \rangle$ is good. Suppose that $\mathbb{R}$ is a forcing such that $\Vdash_{\mathbb{R}}$ '$\langle \mathcal{F}, \mathcal{G} \rangle$ is not good'. Then there is a c.c.c. forcing $\mathbb{Q}$ which adds an uncountable antichain to $\mathbb{R}$.*

First, note that, since $\mathbb{R}_\delta$ is c.c.c. indestructible, it follows by the corollary that $\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$ is good in $V_\alpha^{\mathbb{R}_\delta}$ for $\delta \in \omega_1$, in particular, in V_γ. Now let $e \subseteq \omega$ be an element of V_γ. Since $\mathbb{Q} = \mathbb{R}_{\omega_1}$ is a finite support iteration of c.c.c. posets, e is in some intermediate model $V_\alpha^{\mathbb{R}_\delta}$ for some $\delta \in \omega_1$. Now we claim that $(A_e^\gamma - \delta) \subseteq A_e^\alpha$, where A_e^α is defined in V_γ. If $\xi \in A_e^\gamma - \delta$, then $x_\xi \subseteq^* e$ and $y_\xi \cap e =^* \varnothing$. If there was no $c \in p_\xi^\alpha$ such that $c \subseteq e$, then $p_\xi^\alpha \cup \{\omega - e\}$ would have the finite intersection property and the set $x_\xi \cap (\omega - e)$ would be infinite. Similarly we prove that there is $d \in q_\xi^\alpha$ such that $d \cap e = \varnothing$. Hence $\xi \in A_e^\alpha$ and it follows that A_e^γ is countable. This proves that $\langle \mathcal{F}^\gamma, \mathcal{G}^\gamma \rangle$ is good.

Case 2. α is odd. In this case we take care of all potential c.c.c. forcings. Suppose that $\mathbb{R}$ is a $\mathbb{P}_\alpha$-name for a c.c.c. poset given by some bookkeeping function. If $\Vdash_{\mathbb{R}}$ '$\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$ is good', then we put $\mathbb{Q} = \mathbb{R}$. Assume that $\nVdash_{\mathbb{R}}$ '$\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$ is good'. Since α is odd the elements of $\mathcal{F}^\alpha$ and $\mathcal{G}^\alpha$ are of the form $p_\xi^\alpha = \langle x_\xi \rangle$ and $q_\xi^\alpha = \langle y_\xi \rangle$, where $x_\xi \subseteq^* f$ for all $f \in p_\xi^{\alpha-1}$

and $y_\xi \subseteq^* g$ for all $g \in q_\xi^{\alpha-1}$. Therefore, $\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$ is good if and only if $\langle \{x_\xi : \xi \in \omega_1\}, \{y_\xi : \xi \in \omega_1\} \rangle$ is good. Hence by Lemma 1 there is a c.c.c. forcing $\mathbb{Q}$ such that, in $V_\alpha^{\mathbb{Q}}$, $\mathbb{R}$ is not c.c.c. and $\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$ is good. We force with $\mathbb{Q}$ in this case. Finally, working in V_γ, we define $\langle \mathcal{F}^\gamma, \mathcal{G}^\gamma \rangle$. Since $\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$ is good, by Lemma 1 there is a good pair $\langle \mathcal{F}^\gamma, \mathcal{G}^\gamma \rangle$ that extends $\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$ and such that $\mathcal{F}^\gamma$ and $\mathcal{G}^\gamma$ are families of ultrafilters.

Assume now that γ is a limit. We have to define in V_γ a good pair $\langle \mathcal{F}^\gamma, \mathcal{G}^\gamma \rangle$ which extends all previous pairs. Let $F_\xi = \bigcup\{F_\xi^\alpha : \alpha \in \gamma\}$ and $G_\xi = \bigcup\{G_\xi^\alpha : \alpha \in \gamma\}$, and put $\mathcal{F} = \{F_\xi : \xi \in \omega_1\}$ and $\mathcal{G} = \{G_\xi : \xi \in \omega_1\}$. For $d \subseteq \omega$ let A_d be the set corresponding to $\langle \mathcal{F}, \mathcal{G} \rangle$. We will show that $\langle \mathcal{F}, \mathcal{G} \rangle$ is good in V_γ. Note that this is enough since again by Lemma 1 we can extend $\langle \mathcal{F}, \mathcal{G} \rangle$ to a good pair $\langle \mathcal{F}^\gamma, \mathcal{G}^\gamma \rangle$ such that $\mathcal{F}^\gamma$ and $\mathcal{G}^\gamma$ consist of ultrafilters.

Case 1. $\mathrm{cf}(\gamma) = \omega_1$. Let $d \subseteq \omega$. Then there is an even $\alpha \in \gamma$ such that $d \in V_\alpha$. Since $\mathcal{F}^\alpha$ and $\mathcal{G}^\alpha$ consist of ultrafilters, we have $A_d^\alpha = A_d$, and by the induction hypothesis A_d^α is countable.

Case 2. $\mathrm{cf}(\gamma) = \omega$. Let $d \subseteq \omega$ and suppose that A_d is uncountable. Let $\{\alpha_n : n \in \omega\}$ be a cofinal sequence in γ. Let $A_d^n = \{\xi \in \omega_1 : \exists a \in F_\xi^{\alpha_n}\ \exists b \in G_\xi^{\alpha_n}\ (a \subseteq d \text{ and } b \cap d = \varnothing)\}$. Note that $A_d = \bigcup\{A_d^n : n \in \omega\}$. Let $m \in \omega$ be such that A_d^m is uncountable, and for each $\xi \in A_d^m$ let $a_\xi \in F_\xi^{\alpha_m}$ and $b_\xi \in G_\xi^{\alpha_m}$ be such that $a_\xi \subseteq d$ and $b_\xi \cap d = \varnothing$. Note that $E = \{\langle a_\xi, b_\xi \rangle : \xi \in A_d^m\} \subseteq V^{\mathbb{P}_{\alpha_m}}$. Recall now the following well-known result.

Lemma 3. *Let $\langle \mathbb{R}_n : n \leq \omega \rangle$ be a finite support iteration of c.c.c. posets. Suppose that in $V^{\mathbb{R}_\omega}$ we have an uncountable set E such that $E \subseteq V$. Then there is a $k \in \omega$ and an uncountable set $F \in V^{\mathbb{R}_k}$ such that $F \subseteq E$.*

It follows that there is $k > m$ and an uncountable set $F \in V^{\mathbb{P}_{\alpha_k}}$ such that $F \subseteq E$. Now let $c = \bigcup\{c : \exists d\, \langle c, d \rangle \in F\}$. Obviously $c \in V^{\mathbb{P}_{\alpha_k}}$. Since F is uncountable, it follows that $A_c^{\alpha_k}$ is uncountable, a contradiction. This completes the proof of Theorem 2.

2.1. *Proof of Lemma* 1. Let $\{c_\delta : \delta \in \omega_1\}$ be an enumeration of $\mathcal{P}(\omega)$. By induction on $\alpha \in \omega_1$ we construct $\mathcal{F}^\alpha = \{F_\xi^\alpha : \xi \in \omega_1\}$ and $\mathcal{G}^\alpha = \{G_\xi^\alpha : \xi \in \omega_1\}$ such that the following conditions are satisfied.

(1) $\mathcal{F}^0 = \mathcal{F}$ and $\mathcal{G}^0 = \mathcal{G}$.
(2) For $\xi \in \omega_1$, F_ξ^α and G_ξ^α are filters containing no finite sets.
(3) If $\alpha < \beta$, then $\langle \mathcal{F}^\beta, \mathcal{G}^\beta \rangle$ extends $\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$.
(4) Either c_α or $(\omega - c_\alpha)$ is in $F_\xi^{\alpha+1}$ for $\xi \in \omega_1$ and similarly for $G_\xi^{\alpha+1}$.
(5) $\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$ is good.

We denote by A_e^α the set A_e defined for $\langle \mathcal{F}^\alpha, \mathcal{G}^\alpha \rangle$. Note that if $\alpha < \beta$ and $e \subseteq \omega$, then $A_e^\alpha \subseteq A_e^\beta$. Suppose that $\mathcal{F}^\alpha$ and $\mathcal{G}^\alpha$ have been defined for $\alpha < \gamma$.

Case 1. γ is a limit. Define $F_\xi^\gamma = \bigcup\{F_\xi^\alpha : \alpha < \gamma\}$ and $G_\xi^\gamma = \bigcup\{G_\xi^\alpha : \alpha < \gamma\}$ for all $\xi \in \omega_1$. We have to show that $\langle \mathcal{F}^\gamma, \mathcal{G}^\gamma \rangle$ is good. Let $e \subseteq \omega$. Note that $A_e^\gamma = \bigcup_{\alpha \in \gamma} A_e^\alpha$. Since A_e^α is countable for $\alpha \in \gamma$, it follows that A_e^γ is countable and so $\langle \mathcal{F}^\gamma, \mathcal{G}^\gamma \rangle$ is good.

Case 2. $\gamma = \alpha + 1$. For every $\xi \in \omega_1$ we decide whether $c_\alpha \in F_\xi^\gamma$ or $(\omega - c_\alpha) \in F_\xi^\gamma$ and define $F_\xi^\gamma = \langle F_\xi^\alpha \cup \{c_\alpha\}\rangle$ or $F_\xi^\gamma = \langle F_\xi^\alpha \cup \{\omega - c_\alpha\}\rangle$ respectively. Similarly for G_ξ^γ. To simplify the notation let $F_\xi = F_\xi^\alpha$ and $G_\xi = G_\xi^\alpha$ for $\xi \in \omega_1$ and let $c = c_\alpha$. We have several cases to consider: $\omega_1 = \bigcup\{B_i : i \leq 8\}$ where

$B_0 = \{\xi : c \in F_\xi \text{ and } c \in G_\xi\}$,
$B_1 = \{\xi : c \in F_\xi \text{ and } (\omega - c) \in G_\xi\}$,
$B_2 = \{\xi : (\omega - c) \in F_\xi \text{ and } c \in G_\xi\}$,
$B_3 = \{\xi : (\omega - c) \in F_\xi \text{ and } (\omega - c) \in G_\xi\}$,
$B_4 = \{\xi : c \in F_\xi,\ c \notin G_\xi,\ \text{ and } (\omega - c) \notin G_\xi\}$,
$B_5 = \{\xi : (\omega - c) \in F_\xi \text{ and } c \notin G_\xi \text{ and } (\omega - c) \notin G_\xi\}$,
$B_6 = \{\xi : c \notin F_\xi \text{ and } (\omega - c) \notin F_\xi \text{ and } c \in G_\xi\}$,
$B_7 = \{\xi : c \notin F_\xi \text{ and } (\omega - c) \notin F_\xi,\ (\omega - c) \in G_\xi\}$,
$B_8 = \{\xi : c, (\omega - c) \notin F_\xi \text{ and } c, (\omega - c) \notin G_\xi\}$.

In the first four cases we have nothing to do; we put $F_\xi^\gamma = F_\xi$ and $G_\xi^\gamma = G_\xi$ for all $\xi \in \bigcup\{B_i : i \leq 3\}$. If $\xi \in B_4$, we put $F_\xi^\gamma = F_\xi$ and $G_\xi^\gamma = \langle G_\xi \cup \{c\}\rangle$. If $\xi \in B_5$, we put $F_\xi^\gamma = F_\xi$ and $G_\xi^\gamma = \langle G_\xi \cup \{\omega - c\}\rangle$. If $\xi \in B_6$, we put $F_\xi^\gamma = \langle F_\xi \cup \{c\}\rangle$ and $G_\xi^\gamma = G_\xi$. If $\xi \in B_7$, we put $F_\xi^\gamma = \langle F_\xi \cup \{\omega - c\}\rangle$ and $G_\xi^\gamma = G_\xi$. For $\xi \in B_8$ we extend F_ξ and G_ξ such that $c \in F_\xi^\gamma$ if and only if $c \in G_\xi^\gamma$; i.e., we shall find a function $h : B_8 \to \{0, 1\}$ and put $F_\xi^\gamma = \langle F_\xi \cup \{c^{h(\xi)}\}\rangle$ and $G_\xi^\gamma = \langle G_\xi \cup \{c^{h(\xi)}\}\rangle$, where $c^0 = c$ and $c^1 = \omega - c$. For $\delta \in \omega_1$ we consider sets E_δ^0 and E_δ^1, where for $i \leq 1$

$$E_\delta^i = \{\xi \in B_8 : \exists a \in F_\xi\ \exists b \in G_\xi\ (a \cap c^i \subseteq c_\delta \text{ and } (b \cap c^i) \cap c_\delta = \varnothing)\}.$$

Claim. For every $\delta, \rho \in \omega_1$ the intersection $E_\delta^0 \cap E_\rho^1$ is countable.
Proof. For $\xi \in E_\delta^0 \cap E_\rho^1$ we have $a_\xi^\delta \in F_\xi$ and $b_\xi^\delta \in G_\xi$, $a_\xi^\rho \in F_\xi$ and $b_\xi^\rho \in G_\xi$ as in the definition. Let $a_\xi = a_\xi^\delta \cap a_\xi^\rho$ and $b_\xi = b_\xi^\delta \cap b_\xi^\delta$. Then $a_\xi \in F_\xi$ and $b_\xi \in G_\xi$. Let $e = (c \cap c_\delta) \cup ((\omega - c) \cap c_\rho)$. Note that $a_\xi \subseteq e$ and $b_\xi \subseteq \omega - e$. Therefore, $E_\delta^0 \cap E_\rho^1 \subseteq A_e^\alpha$. Since A_e^α is countable, we are done.

Now we define the function $h : B_8 \to \{0, 1\}$ by induction. Suppose we have defined $h(\xi)$ for all $\xi \in \bigcup\{E_\rho^0 \cup E_\rho^1 : \rho < \delta\}$. For $\xi \in E_\delta^0 - \bigcup\{E_\rho^0 \cup E_\rho^1 : \rho < \delta\}$ we put $h(\xi) = 1$, and for $\xi \in E_\delta^1 - (E_\delta^0 \cup \bigcup\{E_\rho^0 \cup E_\rho^1 : \rho < \delta\})$ we put $h(\xi) = 0$. For $\xi \notin \bigcup\{E_\rho^0 \cup E_\rho^1 : \rho < \omega_1\}$ define $h(\xi)$ arbitrarily. Thus h is everywhere defined and so are F_ξ^γ and G_ξ^γ for all $\xi \in B_8$. Note that by simple induction on $\delta \in \omega_1$, using the Claim, we have

(6) $E_\delta^i \cap h^{-1}(i)$ is countable for $i \leq 1$.

We have to show now that $\langle \mathscr{F}^\gamma, \mathscr{G}^\gamma\rangle$ is good. Let $\delta \in \omega_1$. To show that $|A_{c_\delta}^\gamma| < \aleph_1$ it is enough to show that $|A_{c_\delta}^\gamma \cap B_i| < \aleph_1$ for $i \leq 8$. This is easy for $i < 8$, so we only show that $|A_{c_\delta}^\gamma \cap B_8| < \aleph_1$. Note that by (6) it is enough to show that

$$A_{c_\delta}^\gamma \cap h^{-1}(i) \subseteq E_\delta^i \cap h^{-1}(i)$$

for $i \leq 1$. Recall that $A_{c_\delta}^\gamma = \{\xi : \exists \bar{a} \in F_\xi^\gamma\ \exists \bar{b} \in G_\xi^\gamma\ \bar{a} \subseteq c_\delta,\ \bar{b} \cap c_\delta = \varnothing\}$. Let $\xi \in A_{c_\delta}^\gamma \cap h^{-1}(i)$. Then there are $\bar{a} \in F_\xi^\gamma$ and $\bar{b} \in G_\xi^\gamma$ as in the definition of $A^{\gamma}_{c_\delta}$. Moreover, $\bar{a} = a \cap c^i$ and $\bar{b} = b \cap c^i$ for some $a \in F_\xi$ and $b \in G_\xi$. Hence $a \cap c^i \subseteq c_\delta$ and $(b \cap c^i) \cap c_\delta = \varnothing$ and so $\xi \in E^i_\delta \cap h^{-1}(i)$.

Finally define $\overline{\mathscr{F}} = \mathscr{F}^{\omega_1}$ and $\overline{\mathscr{G}} = \mathscr{G}^{\omega_1}$. It follows from (2)–(4) that $\overline{\mathscr{F}}$ and $\overline{\mathscr{G}}$ are families of ultrafilters. Let $c \subseteq \omega$ and let $\gamma \in \omega_1$ be such that $c = c_\gamma$. Thus c has been considered at the stage $\gamma + 1$. Since $\langle \mathscr{F}^{\gamma+1}, \mathscr{G}^{\gamma+1} \rangle$ is good, it follows that $A^{\gamma+1}_c$ is countable. Moreover, (4) implies that $A^{\gamma+1}_c = A^{\omega_1}_c$. Hence $A^{\omega_1}_c$ is countable and $\langle \overline{\mathscr{F}}, \overline{\mathscr{G}} \rangle$ is good. This completes the proof of Lemma 1.

2.2. *Proof of Lemma* 2. We use the method developed in [1] to construct $\mathbb{Q}$. Fix $\{r_\alpha : \alpha \in \omega_1\}$ to be an enumeration of the reals. Let $[x_\xi] = \{a \subseteq \omega : a =^* x_\xi\}$ and $[y_\xi] = \{b \subseteq \omega : b =^* y_\xi\}$, and let $\{\langle a_\xi, b_\xi \rangle : \xi \in \omega_1\}$ be an enumeration of the set $\bigcup\{[x_\alpha] \times [y_\alpha] : \alpha \in \omega_1\}$.

Fix an increasing and continuous sequence of elementary submodels $\{N_\alpha \prec H(\omega_3) : \alpha \in \omega_1\}$ such that $\{r_\alpha : \alpha \in \omega_1\}$, $\{\langle a_\xi, b_\xi \rangle : \xi \in \omega_1\} \in N_0$, and $\{N_\alpha \cap \omega_1 : \alpha \in \omega_1\}$ is a closed unbounded subset of ω_1. Let $C = \{\alpha \in \omega_1 : \alpha = N_\alpha \cap \omega_1\}$. Then C is also a club in ω_1. Note that for every real, r there is $\gamma \in C$ such that $r \in N_\gamma$. Let $\delta \in \omega_1$ and $\{t_\alpha : \alpha \in \delta\}$ be any sequence of finite subsets of ω_1. Since it can be coded as a real, it follows that there is $\gamma \in C$ such that $\{t_\alpha : \alpha \in \delta\} \in N_\gamma$.

Since $\langle F, G \rangle$ is not good in $V^{\mathbb{R}}$, there must be a $p \in \mathbb{R}$ and an $\mathbb{R}$-name d such that $p \Vdash |A_d| = \aleph_1$. By induction on $\alpha \in \omega_1$ we define $\langle p_\alpha, \alpha_0, \alpha_1 \rangle \in \mathbb{R} \times \omega_1 \times \omega_1$ such that

(1) there are $c_0, c_1 \in C$ such that $c_0 < \alpha_0 < c_1 < \alpha_1$,
(2) if $\alpha < \beta$, then $\alpha_1 < c < \beta_0$ for some $c \in C$, and
(3) $p_\alpha \Vdash$ '$(a_{\alpha_0} \cup a_{\alpha_1}) \subseteq d$ and $(b_{\alpha_0} \cup b_{\alpha_1}) \cap d = \varnothing$'.

Suppose that $\langle p_\alpha, \alpha_0, \alpha_1 \rangle$ have been defined for all $\alpha < \beta$. Let $c \in C$ be such that $c > \sup\{\alpha_1 : \alpha \in \beta\}$. Since $p \Vdash |A_d| = \aleph_1$, there are $p^0_\beta \le p$ and $\beta_0 > c$ such that $p^0_\beta \Vdash$ '$(a_{\beta_0} \subseteq d)$ and $(b_{\beta_0} \cap d = \varnothing)$'. By repeating the construction we can find $p_\beta \le p^0_\beta$ and $\beta_1 > c > \beta_0$ for some $c \in C$ such that (1)–(3) are satisfied for all $\alpha \le \beta$. This finishes the induction step. Now we define $\mathbb{Q}$ to be a set of finite approximations to an antichain in $\{p_\alpha : \alpha \in \omega_1\}$: the set of finite sets s that satisfy: if $\alpha \ne \beta$ in s, then

$$(a_{\alpha_0} \cap b_{\beta_0}) \cup (a_{\beta_0} \cap b_{\alpha_0}) \ne \varnothing \quad \text{or} \quad (a_{\alpha_1} \cap b_{\beta_1}) \cup (a_{\beta_1} \cap b_{\alpha_1}) \ne \varnothing.$$

$\mathbb{Q}$ is ordered by reverse inclusion. Note that if $\alpha, \beta \in s$ for some $s \in \mathbb{Q}$, then p_α and p_β are incompatible. Therefore, if G is $\mathbb{Q}$-generic, then $\{p_\alpha : \alpha \in \bigcup G\}$ is an antichain in $\mathbb{R}$.

Sublemma. *Let $k \in \omega$, and let $\{t_\alpha : \alpha \in \omega_1\}$ be a sequence of disjoint k-tuples $t_\alpha = \{\alpha^1, \dots, \alpha^k\} \subseteq \omega_1$ such that $\alpha^1 < c_1 < \cdots < c_{k-1} < \alpha^k$ for some $c_1, \dots, c_{k-1} \in C$. Then there are $\alpha, \beta \in \omega_1$ such that*

$$(a_{\alpha^i} \cap b_{\beta^i}) \cup (a_{\beta^i} \cap b_{\alpha^i}) \ne \varnothing \quad \textit{for } i \le k.$$

Proof. Let $\delta \in \omega_1$ be such that for every sequence $\langle n_1, m_1, \dots, n_k, m_k \rangle$ of elements of $\omega \cup \{-1\}$, if there is an $\alpha \in \omega_1$ such that $n_i \in a_{\alpha^i}$ and $m_i \in b_{\alpha^i}$ for $i \le k$, then there is $\beta < \delta$ such that $n_i \in a_{\beta^i}$ and $m_i \in b_{\beta^i}$ for $i \le k$, where we assume $-1 \in a$ to be always true. (We use -1 to simplify the notation. Sometimes we will consider incomplete sequences, e.g., $\langle n_1, m_1, \dots, n_{k-1}, n_k, m_k\rangle$, but we can always write it as a $2k$-tuple $\langle n_1, \dots, n_{k-1}, -1, n_k, m_k\rangle$.)

Let $\gamma \in C$ be such that $\{t_\zeta : \zeta < \delta\} \in N_\gamma$ and pick $\alpha \in \omega_1$ such that $\gamma < \alpha^1 < c_1 < \cdots < c_{k-1} < \alpha^k$ with $c_1, \dots, c_{k-1} \in C$. For $x \in \omega_1$ we consider the following statement:

$$\begin{array}{ll} \Phi_k(x): & \underline{\text{For every}}\ \langle n_1, m_1, \dots, n_k, m_k\rangle \in^{2k} (\omega \cup \{-1\})\ \textit{such that} \\ & \quad (1)\ n_i \in a_{\alpha^i}\ \textit{and}\ m_i \in b_{\alpha^i}\ \textit{for}\ i \le k-1\ \textit{and} \\ & \quad\ n_k \in a_x\ \textit{and}\ m_k \in b_x \\ & \underline{\text{there is}}\ \tau < \delta\ \textit{such that}\ n_i \in a_{\tau^i}\ \textit{and}\ m_i \in b_{\tau^i}\ \textit{for}\ i \le k. \end{array}$$

Note that $\Phi_k(\alpha^k)$ holds by the definition of δ. Moreover, since all parameters of Φ_k are in $N_{c_{k-1}}$ and $\alpha^k > c_{k-1}$, it follows that the set $W_k = \{x \in \omega_1 : \Phi_k(x)\}$ is uncountable. We claim that there are $\xi_k, \eta_k \in W_k$ such that $a_{\xi_k} \cap b_{\eta_k} \neq \varnothing$. Indeed, recall that $[x_\xi] \times [y_\xi]$ is countable for $\xi \in \omega_1$. Therefore, if we had $(\bigcup\{a_\xi : \xi \in W_k\}) \cap (\bigcup\{b_\xi : \xi \in W_k\}) = \varnothing$, then A_e would be uncountable, where $e = \bigcup\{a_\xi : \xi \in W_k\}$. Since $\langle F, G\rangle$ is good, this is impossible.

Let $z_k \in a_{\xi_k} \cap b_{\eta_k}$. By downward induction on $1 \le i \le k$ we define formulas $\Phi_i(x)$ together with $z_i \in \omega$ and ξ_i and η_i in ω_1 such that $z_i \in a_{\xi_i} \cap b_{\eta_i}$ and $\Phi_i(\alpha^i)$, $\Phi_k(\xi_i)$, and $\Phi_i(\eta_i)$ hold. Suppose that Φ_i, z_i, ξ_i, and η_i, have been already defined for $l < i \le k$. Define $\Phi_l(x)$ by

$$\begin{array}{ll} \Phi_l(x): & \underline{\text{For every}}\ \langle n_1, m_1, \dots, n_l, m_l\rangle \in^{2l} (\omega \cup \{-1\})\ \textit{such that} \\ & \quad (1)\ n_i \in a_{\alpha^i}\ \textit{and}\ m_i \in b_{\alpha^i}\ \textit{for}\ 1 \le i < l\ \textit{and} \\ & \quad\ n_l \in a_x\ \textit{and}\ m_l \in b_x \\ & \underline{\text{there are}}\ \tau, \sigma < \delta\ \textit{such that} \\ & \quad (2)\ n_i \in a_{\tau^i}\ \textit{and}\ m_i \in b_{\tau^i}\ \textit{for}\ 1 \le i \le l\ \textit{and} \\ & \quad\ z_j \in a_{\tau^j}\ \textit{for}\ l < j \le k,\ \textit{and} \\ & \quad (3)\ n_i \in a_{\sigma^i}\ \textit{and}\ m_i \in b_{\sigma^i}\ \textit{for}\ 1 \le i \le l\ \textit{and} \\ & \quad\ z_j \in b_{\sigma^j}\ \textit{for}\ l < j \le k. \end{array}$$

We show that $\Phi_l(\alpha^l)$ holds. Let $\langle n_1, m_1, \dots, n_l, m_l\rangle \in^{2l} (\omega \cup \{-1\})$ be any sequence which satisfies condition (1) of the statement $\Phi_l(\alpha^l)$. Note that $\langle n_1, \dots, m_l, z_{l+1}, -1\rangle$ satisfies condition (1) of the statement $\Phi_{l+1}(\xi_{l+1})$. Therefore, there are $\tau_1, \sigma_1 < \delta$ ($\tau < \delta$ if $l = 1$) as in $\Phi_{l+1}(\xi_{l+1})$. Similarly $\langle n_1, \dots, m_l, -1, z_{l+1}\rangle$ satisfies condition (1) of $\Phi_{l+1}(\eta_{l+1})$. Let $\tau_2, \sigma_2 < \delta$ ($\sigma < \delta$ if $l = 1$) be as in the conclusion of $\Phi_{l+1}(\eta_{l+1})$. Now define $\tau = \tau_1$ and $\sigma = \sigma_2$. It is easy to see that τ and σ satisfy conditions (2) and (3) of the statement $\Phi_l(\alpha^l)$. Since all parameters of Φ_l are in $N_{c_{l-1}}$ and $\alpha^l > c_{l-1}$, it follows that the set $W_l = \{x \in \omega_1 : \Phi_l(x)\}$ is uncountable. Therefore, there are $\xi_l, \eta_l \in \omega_1$ and $z_l \in \omega$ such that $z_l \in a_{\xi_l} \cap b_{\eta_l}$ and $\Phi_l(\xi_l)$ and $\Phi_l(\eta_l)$ hold. This finishes the induction step.

Finally using the sequence $\langle z_1, -1\rangle$ and the statement $\Phi_1(\xi_1)$ we can obtain τ_1 and $\sigma_1 < \delta$ and similarly $\langle -1, z_1\rangle$ together with $\Phi_1(\eta_1)$ gives τ_2 and σ_2. Let $\alpha = \tau_1$ and $\beta = \sigma_2$. It follows from the definition of Φ_1 that α and β satisfy the requirements of the sublemma, i.e., $z_i \in a_{\alpha^i} \cap b_{\beta^i}$ for $i = 1 \le k$.

We are ready to prove that $\mathbb{Q}$ satisfies the c.c.c. Let $\{s_\alpha : \alpha \in \omega_1\}$ be an uncountable subset of $\mathbb{Q}$. Thinning out if necessary we can assume that

(4) there is $k \in \omega$ such that $|s_\alpha| = k$ and $s_\alpha = \{\alpha^1, \dots, \alpha^k\}$,
(5) $s_\alpha \cap s_\beta = \varnothing$ for $\alpha \neq \beta$, and
(6) if $\alpha, \beta \in \omega_1$, then $a_{\alpha_i^l} \cap b_{\alpha_j^r} = a_{\beta_i^l} \cap b_{\beta_j^r}$ for $l, r \leq k$ and $i, j \leq 1$.

Let $t_\alpha = \{\alpha_0^1, \dots, \alpha_0^k\}$. Note that $\{t_\alpha : \alpha \in \omega_1\}$ satisfies the assumptions of the sublemma. Therefore, there are $\alpha, \beta \in \omega_1$ such that $(a_{\alpha_0^l} \cap b_{\beta_0^l}) \cup (a_{\beta_0^l} \cap b_{\alpha_0^l}) \neq \varnothing$ for $l \leq k$. This together with (6) implies that s_α and s_β are compatible and hence $\mathbb{Q}$ is c.c.c.

To show that $\langle F, G\rangle$ is good in $V^{\mathbb{Q}}$ we modify the proof of the fact that $\mathbb{Q}$ is c.c.c. Suppose that $\langle F, G\rangle$ is not good in $V^{\mathbb{Q}}$. Then there is a $\mathbb{Q}$-name e and a sequence $\{\langle s_\alpha, \bar{\alpha}\rangle : \alpha \in \omega_1\}$ such that $s_\alpha \Vdash$ '$(a_{\bar{\alpha}} \subseteq e)$ and $(b_{\bar{\alpha}} \cap e = \varnothing)$' and such that $\bar{\alpha} > \bar{\beta}$ for $\alpha > \beta$. As before we can assume that (4)–(6) are satisfied. Recall that for every $\alpha \in \omega_1$ there are $\{c_j \in C : 1 \leq j \leq 2k+1\}$ such that $c_{2l+j-1} < \alpha_j^l < c_{2l+j}$ for $l \leq k$ and $j \leq 1$, i.e., α_0^l and α_1^l are in disjoint intervals with endpoints in C. For each l let $h(l) \in \{0, 1\}$ be such that $\alpha_{h(l)}^l$ and $\bar{\alpha}$ are not in the same interval.

Define $(k+1)$-tuples $t_\alpha = \{\alpha_{h(1)}^1, \dots, \alpha_{h(k)}^k, \bar{\alpha}\}$. Without loss of generality we can assume that $\{t_\alpha : \alpha \in \omega_1\}$ form a Δ-system and the root is empty. Moreover, we can assume that the place of $\bar{\alpha}$ in the sequence t_α does not depend on α, e.g., $\bar{\alpha}$ is the last element of t_α. Now $\{t_\alpha : \alpha \in \omega_1\}$ satisfy the assumptions of the sublemma; hence, there are $\alpha, \beta \in \omega_1$ such that

(7) $(a_{\bar{\alpha}} \cap b_{\bar{\beta}}) \cup (a_{\bar{\beta}} \cap b_{\bar{\alpha}}) \neq \varnothing$ and
(8) $(a_{\alpha_{h(l)}^l} \cap b_{\beta_{h(l)}^l}) \cup (a_{\beta_{h(l)}^l} \cap b_{\alpha_{h(l)}^l}) \neq \varnothing$ for $l \leq k$.

Note that (8) and (6) imply that s_α and s_β are compatible. Let $s \leq s_\alpha, s_\beta$; then $s \Vdash$ '$(a_{\bar{\alpha}} \cup a_{\bar{\beta}}) \subseteq e$ and $(b_{\bar{\alpha}} \cup b_{\bar{\beta}}) \cap e = \varnothing$'. This contradicts (7).

To finish the proof note that, since $\mathbb{Q}$ is a c.c.c. poset, we can find a condition $s \in \mathbb{Q}$ which forces that the generic G is uncountable. Hence s forces that $\{p_\alpha : \alpha \in \bigcup G\}$ is an uncountable antichain in $\mathbb{R}$. This completes the proof.

2.3. *Proof of the corollary*. This is very similar to the proof of Lemma 1. We point out only the main differences. Note that we do not claim now that $\langle \mathscr{F}, \mathscr{G}\rangle$ is good in the extension by $\mathbb{Q}$. Let $\{\langle p_\alpha, \bar{\alpha}\rangle : \alpha \in \omega_1\}$ and an $\mathbb{R}$-name d be such that for every α there are $\gamma(\alpha) < \omega_1$, $a_{\bar{\alpha}} \in F_{\gamma(\alpha)}$, and $b_{\bar{\alpha}} \in G_{\gamma(\alpha)}$ satisfying

(1) $\gamma(\alpha) < \gamma(\beta)$ for $\alpha < \beta$ and
(2) $p_\alpha \in \mathbb{R}$ and $p_\alpha \Vdash$ '$a_{\bar{\alpha}} \subseteq d$ and $b_{\bar{\alpha}} \cap d = \varnothing$'.

Let $\{M_\alpha : \alpha \in \omega_1\}$ be an increasing, continuous sequence of elementary submodels of $H(\omega_3)$ such that $\{\langle a_{\bar{\alpha}}, b_{\bar{\alpha}}\rangle : \alpha \in \omega_1\}$ and $\{\bar{\alpha} : \alpha \in \omega_1\}$ are in M_0. Let $C \in \omega_1$ be a club in ω_1 given by $\{M_\alpha : \alpha \in \omega_1\}$. Define $\mathbb{Q}$ to be the set of those finite subsets of $\{\bar{\alpha} : \alpha \in \omega_1\}$ that are separated by C and that satisfy if $\alpha \neq \beta$ then $(a_{\bar{\alpha}} \cap b_{\bar{\beta}}) \cup (a_{\bar{\beta}} \cap b_{\bar{\alpha}}) \neq \varnothing$. Order $\mathbb{Q}$ by reverse inclusion. A suitable form of the sublemma implies that $\mathbb{Q}$ is c.c.c. If G is a $\mathbb{Q}$-generic set, then $\{p_\alpha : \bar{\alpha} \in \bigcup G\}$ is an antichain in $\mathbb{R}$. As before we can assume that this is an uncountable antichain. This completes the proof.

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