AN INEQUALITY FOR SECTIONS AND PROJECTIONS OF A CONVEX SET

JONATHAN E. SPINGARN

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ABSTRACT. Let $K \subset \mathbf{R}^d$ be a convex body, γ its center of mass. For $\Lambda \subset \mathbf{R}^d$ a subspace of dimension $d - k$, we establish the inequality

$$\text{Vol}_d(K) \leq \text{Vol}_{d-k}(K|\Lambda) \text{Vol}_k((K - \gamma) \cap \Lambda^\perp)$$

(where $K|\Lambda$ denotes orthogonal projection of K onto Λ). Equality holds only if each k -dimensional section of K parallel to $\Lambda^\perp$ is a translate of $(K - \gamma) \cap \Lambda^\perp$.

1. SETTING AND STATEMENT OF RESULT

Let $K \subset \mathbf{R}^d$ be a convex body (a compact convex set with interior), γ its center of mass, and Λ a subspace of $\mathbf{R}^d$ of dimension $d - k$. The purpose of this note is to establish that

$$(1) \quad \text{Vol}_d(K) \leq \text{Vol}_{d-k}(K|\Lambda) \text{Vol}_k((K - \gamma) \cap \Lambda^\perp),$$

$K|\Lambda$ denoting the orthogonal projection of K onto Λ . Furthermore, equality holds only if each k -dimensional section of K parallel to $\Lambda^\perp$ is a translate of $(K - \gamma) \cap \Lambda^\perp$.

An inequality in the opposite sense,

$$\binom{d}{k} \text{Vol}_d(K) \geq \text{Vol}_{d-k}(K|\Lambda) \text{Vol}_k((K - \zeta) \cap \Lambda^\perp)$$

(for arbitrary $\zeta \in K$) has been obtained by Rogers and Shephard [11] and also Chakerian [3]. Lutwak [8] has observed that an inequality which replaces the intersection term in (1) by an average over all slices parallel to $\Lambda^\perp$,

$$\text{Vol}_d(K) \leq \frac{\text{Vol}_{d-k}(K|\Lambda)}{\text{Vol}_d(K)} \int_K \text{Vol}_k((K - x) \cap \Lambda^\perp) d\mu_d(x),$$

follows easily from the Cauchy-Schwarz inequality. (Here, μ_d denotes Lebesgue measure on $\mathbf{R}^d$.)

2. PROOF OF THE INEQUALITY

Consider the mapping $\Phi(x) = (K - x) \cap \Lambda^\perp$, whose values are compact convex sets in the k -dimensional subspace $\Lambda^\perp$. Since K is convex, Φ is a

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concave set-valued mapping in the sense that, for all $x, y \in \mathbf{R}^d$ and $0 \leq t \leq 1$, $t\Phi(x) + (1-t)\Phi(y) \subset \Phi(tx + (1-t)y)$. To see this, note that

$$\begin{aligned} t\Phi(x) + (1-t)\Phi(y) &= t((K-x) \cap \Lambda^\perp) + (1-t)((K-y) \cap \Lambda^\perp) \\ &\subset (K - (tx + (1-t)y)) \cap \Lambda^\perp \\ &= \Phi(tx + (1-t)y). \end{aligned}$$

For all x for which $\Phi(x)$ is nonempty (that is, for $x \in K + \Lambda^\perp$), let $r(x)$ denote the radius of a k -ball whose k -dimensional volume equals that of $\Phi(x)$. According to the Brunn-Minkowski Theorem [1, p. 88], the concavity of Φ implies that r is a concave function,

$$(2) \quad tr(x) + (1-t)r(y) \leq r(tx + (1-t)y) \quad (x, y \in K + \Lambda^\perp, 0 < t < 1),$$

with equality if and only if

$$(3) \quad \begin{aligned} &\text{either } t\Phi(x) + (1-t)\Phi(y) = \Phi(tx + (1-t)y) \\ &\quad \text{and } \Phi(x) \text{ and } \Phi(y) \text{ are homothetic,} \\ &\text{or } \Phi(x) \text{ and } \Phi(y) \text{ lie in parallel hyperplanes of } \Lambda^\perp \\ &\quad \text{(in which case both sides are zero).} \end{aligned}$$

We now obtain a set K_0 by “symmetrizing” K with respect to Λ . Define $K_0 = \{x + u : x \in K|_\Lambda, u \in \Lambda^\perp, |u| \leq r(x)\}$, and let γ_0 denote the center of mass of K_0 . If we can establish that K_0 is convex and satisfies the properties:

$$(4) \quad \begin{aligned} (i) \quad &K|_\Lambda = K_0|_\Lambda, \\ (ii) \quad &\text{Vol}_d(K) = \text{Vol}_d(K_0), \\ (iii) \quad &\text{Vol}_k((K - \gamma) \cap \Lambda^\perp) = \text{Vol}_k((K_0 - \gamma_0) \cap \Lambda^\perp), \end{aligned}$$

it will be immediate that K_0 satisfies (1) if and only if K does.

To see that K_0 is convex, suppose that $w_1, w_2 \in K_0$ and $0 \leq \lambda \leq 1$. It is possible to write

$$w_i = x_i + u_i, \quad x_i \in K|_\Lambda, \quad u_i \in \Lambda^\perp, \quad |u_i| \leq r(x_i) \quad (i = 1, 2).$$

Then $\lambda w_1 + (1-\lambda)w_2 = (\lambda x_1 + (1-\lambda)x_2) + (\lambda u_1 + (1-\lambda)u_2)$. Now $\lambda x_1 + (1-\lambda)x_2 \in K|_\Lambda$ and, by the concavity of r , $|\lambda u_1 + (1-\lambda)u_2| \leq \lambda r(x_1) + (1-\lambda)r(x_2) \leq r(\lambda x_1 + (1-\lambda)x_2)$. Thus $\lambda w_1 + (1-\lambda)w_2 \in K_0$, showing that K_0 is a convex set. The property (i) is immediate from the definition of K_0 . For each $x \in K|_\Lambda$, $(K_0 - x) \cap \Lambda^\perp$ is a k -ball of radius $r(x)$ whose k -dimensional volume is the same as that of $\Phi(x) = (K - x) \cap \Lambda^\perp$ (both equal to $\omega_k r(x)^k$, where ω_k denotes the volume of the Euclidean unit ball in $\mathbf{R}^k$). So,

$$\text{Vol}_d(K_0) = \int_{K_0|_\Lambda} \omega_k r(x)^k d\mu_{d-k}(x) = \int_{K|_\Lambda} \text{Vol}_k(\Phi(x)) d\mu_{d-k}(x) = \text{Vol}_d(K),$$

which is property (ii). Let γ_0 be the center of mass of K_0 . Since K_0 is symmetric with respect to Λ , we know that $\gamma_0 \in \Lambda$ and

$$\begin{aligned} \gamma_0 &= \frac{1}{\text{Vol}_d(K_0)} \int_{K_0|_\Lambda} x \omega_k r(x)^k d\mu_{d-k}(x) \\ &= \frac{1}{\text{Vol}_d(K)} \int_{K|_\Lambda} x \text{Vol}_k(\Phi(x)) d\mu_{d-k}(x) = \gamma|_\Lambda. \end{aligned}$$

In particular, $(K - \gamma) \cap \Lambda^\perp$ is a translate of $(K - \gamma_0) \cap \Lambda^\perp$, so

$$\text{Vol}_k((K - \gamma) \cap \Lambda^\perp) = \text{Vol}_k((K - \gamma_0) \cap \Lambda^\perp) = \text{Vol}_k((K_0 - \gamma_0) \cap \Lambda^\perp),$$

which is property (iii). Since (4) holds, it is clear that K_0 satisfies (1) if and only if K does. Therefore, we will lose no generality if we henceforth assume K to be symmetric with respect to Λ ; that is, $K = K_0$ and $\gamma = \gamma_0$.

Let $1 < p < \infty$, $1/p + 1/q = 1$, and let L_p and L_q denote the p - and q -summable functions (with respect to an arbitrary measure space). Hölder's inequality asserts that if $f \in L_p$ and $g \in L_q$ then $fg \in L_1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$ with equality if and only if

$$(5) \quad \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} = \frac{1}{p} \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g|^q}{\|g\|_q^q}$$

almost everywhere [7]. Applying Hölder's inequality with $f = 1$, $g = r(x)^k$, $p = k + 1$, $q = (k + 1)/k$, we obtain

$$(6) \quad \int_{K|\Lambda} r(x)^k d\mu_{d-k} \leq \text{Vol}_{d-k}(K|\Lambda)^{1/(k+1)} \left(\int_{K|\Lambda} r(x)^{k+1} d\mu_{d-k} \right)^{k/(k+1)}.$$

Since r is concave and $\gamma \in \text{relint}(K|\Lambda)$, there exists $z \in \Lambda$ with

$$r(x) \leq r(\gamma) + \langle z, x - \gamma \rangle \quad (x \in K|\Lambda)$$

[10, Theorem 23.4]. Since γ is the center of mass of K , we have $\int r(x)^k (x - \gamma) d\mu_{d-k} = 0$, whence

$$(7) \quad \begin{aligned} \int_{K|\Lambda} r(x)^{k+1} d\mu_{d-k} &\leq \int_{K|\Lambda} r(x)^k (r(\gamma) + \langle z, x - \gamma \rangle) d\mu_{d-k} \\ &= r(\gamma) \int_{K|\Lambda} r(x)^k d\mu_{d-k}. \end{aligned}$$

Combining (6) and (7), we obtain

$$\int_{K|\Lambda} r(x)^k d\mu_{d-k} \leq \text{Vol}_{d-k}(K|\Lambda)^{1/(k+1)} r(\gamma)^{k/(k+1)} \left(\int_{K|\Lambda} r(x)^k d\mu_{d-k} \right)^{k/(k+1)},$$

which is equivalent to

$$\int_{K|\Lambda} \omega_k r(x)^k d\mu_{d-k} \leq \text{Vol}_{d-k}(K|\Lambda) \omega_k r(\gamma)^k,$$

which is (1).

Next we examine the conditions for equality in (1). It follows from the compactness of K that Φ is *upper semicontinuous* in the sense that for every x and $\varepsilon > 0$ there exists δ such that $|y - x| < \delta$ implies $\Phi(y) \subset \Phi(x) + \varepsilon B$ (where B denotes the unit ball in $\Lambda^\perp$). Consequently, r is upper semicontinuous, in the sense that for any convergent sequence $x_n \in K + \Lambda^\perp$ we have

$$(8) \quad \limsup r(x_n) \leq r(x) \quad (x_n \rightarrow x)$$

[6, Chapter 4.4.1]. Now, r may fail to be actually continuous on $K + \Lambda^\perp$, but being concave, r must be continuous on $\text{relint}(K + \Lambda^\perp)$ [10, Theorem 10.1]. In particular, r is continuous on $\text{relint}(K|\Lambda)$.

If equality holds in (1), then it must hold also in (6). From (5) and the continuity of r on $\text{relint}(K + \Lambda^\perp)$, equality in (6) would imply the existence of positive numbers α and β such that $r(x)^k = \alpha + \beta r(x)^{k+1}$ for all $x \in \text{relint}(K|\Lambda)$. But the straight line in $\mathbf{R}^2$ whose equation is $y = \alpha + \beta t$ can intersect the curve $y = t^{k/(k+1)}$ in at most two points. This means that $r(x)$ can attain at most two values on $\text{relint}(K|\Lambda)$, so that, by continuity, $r(x)$ is actually equal to a constant value α on $\text{relint}(K|\Lambda)$. We show next, however, that if $r \equiv \alpha$ on $\text{relint}(K|\Lambda)$, then actually $r \equiv \alpha$ on $K|\Lambda$. To see this, pick $x_0 \in \text{relint}(K|\Lambda)$ and suppose $x \in \text{relbd}(K|\Lambda)$. Fix any $y_0 \in \Phi(x_0)$. For any $t \in (0, 1)$, $\Phi((1-t)x_0 + tx) \supset (1-t)y_0 + t\Phi(x)$. Therefore,

$$\begin{aligned}\omega_k \alpha^k &= \text{Vol}_k(\Phi((1-t)x_0 + tx)) \geq \text{Vol}_k((1-t)y_0 + t\Phi(x)) \\ &= t^k \text{Vol}_k(\Phi(x)) = t^k \omega_k r(x)^k.\end{aligned}$$

Letting $t \rightarrow 1$, we see that $r(x) \leq \alpha$. Combining this observation with (8), we see that $r(x) = \alpha$, so $r \equiv \alpha$ on $K|\Lambda$. Equality holds therefore in (2) (for arbitrary $x, y \in K|\Lambda$ and $0 < t < 1$) so that the conditions (3) must also be satisfied. Thus for every $x, y \in K|\Lambda$, $\Phi(x)$ and $\Phi(y)$ are homothetic. (The other alternative, that $\Phi(x)$ and $\Phi(y)$ lie in parallel hyperplanes of $\Lambda^\perp$, cannot hold since this would imply all sections of K parallel to $\Lambda^\perp$ have zero k -dimensional volume.) Since $\Phi(x)$ and $\Phi(y)$ have equal volume, they are identical modulo a translation.

3. REMARKS

The inequality (1) has some immediate consequences pertaining to the “quermassintegrals” of K . Let Γ denote the Grassman manifold of $(d-k)$ -dimensional subspaces of $\mathbf{R}^d$, and let η denote the usual Haar measure on Γ , normalized so that $\eta(\Gamma) = 1$. Dividing both sides of (1) by $\text{Vol}_{d-k}(K|\Lambda)$ and integrating over all $\Lambda \in \Gamma$,

$$(9) \quad \text{Vol}_d(K) \int_{\Gamma} \frac{d\eta(\Lambda)}{\text{Vol}_{d-k}(K|\Lambda)} \leq \int_{\Gamma} \text{Vol}_k((K - \gamma) \cap \Lambda^\perp) d\eta(\Lambda).$$

Let us consider the circumstances under which equality holds in (9). It holds trivially when $k = 0$ or $k = d$, so we assume $0 < k < d$ and consider first the case when $k = d - 1$. Since the integrands depend continuously on Λ , equality in (9) requires equality of the integrands and, hence, equality in (1) for all $\Lambda \in \Gamma$. Thus, for every $\Lambda \in \Gamma$, each $(d-1)$ -dimensional section of K parallel to $\Lambda^\perp$ is a translate of $(K - \gamma) \cap \Lambda^\perp$. A convex body for which there are $d-1$ linearly independent directions, in each of which all the $(d-1)$ -dimensional slices have the same $(d-1)$ -dimensional volume, is a d -dimensional parallelotope [5]. But as a parallelotope K cannot have the property: for all $\Lambda \in \Gamma$, each $(d-1)$ -dimensional section of K parallel to $\Lambda^\perp$ is a translate of $(K - \gamma) \cap \Lambda^\perp$. This contradiction shows the inequality (9) to be strict when $k = d - 1$. If $k < d - 1$, we may use the result from [5] in a similar fashion, combined with Fubini's theorem, to show that equality cannot hold in (9). The inequality (9) is therefore strict, except when $k = 0, d$.

Hadwiger's *harmonic quermassintegrals* $\widehat{W}_k(K)$ are defined by

$$\frac{1}{\widehat{W}_k(K)} = \frac{\omega_{d-k}}{\omega_d} \int_{\Gamma} \frac{d\eta(\Lambda)}{\text{Vol}_{d-k}(K|\Lambda)},$$

when $0 < k < d$, and $\widehat{W}_0(K) = \text{Vol}_d(K)$, $\widehat{W}_d(K) = \omega_d$ [6]. The $(d - k)$ th dual quermassintegral $\widetilde{W}_{d-k}(K - \gamma)$ of $K - \gamma$ is proportional to the mean of the k -dimensional volumes of the slices of $K - \gamma$ by the k -dimensional subspaces of $\mathbf{R}^d$ [9]. Specifically, for $0 < k < d$,

$$\widetilde{W}_{d-k}(K - \gamma) = \frac{\omega_d}{\omega_k} \int_{\Gamma} \text{Vol}_k((K - \gamma) \cap \Lambda^\perp) d\eta(\Lambda),$$

while $\widetilde{W}_0(K - \gamma) = \text{Vol}_d(K)$ and $\widetilde{W}_d(K - \gamma) = \omega_d$. Therefore, it is possible to rewrite (9) to obtain an inequality relating the k th harmonic quermassintegral to the complementary dual quermassintegral:

$$\widetilde{W}_{d-k}(K - \gamma) \widehat{W}_k(K) \geq \text{Vol}_d(K) \frac{\omega_d^2}{\omega_k \omega_{d-k}},$$

the inequality being strict when $0 < k < d$. (Since the dual quermassintegral is not translation invariant, the fact that $\widetilde{W}_{d-k}$ is computed here with respect to γ as origin is significant.) A related inequality

$$\overline{W}_{d-k}(K) \widehat{W}_k(K) \geq \text{Vol}_d(K) \frac{\omega_d^2}{\omega_k \omega_{d-k}}$$

(strict when $k \neq 0, d$) has been obtained by Lutwak [8], where

$$\overline{W}_i(K) = \frac{1}{\text{Vol}_d(K)} \int_K \widetilde{W}_i(K - x) d\mu_d(x)$$

is the mean of the i th dual quermassintegral.

In a similar manner, we can derive an inequality relating to the affine and dual affine quermassintegrals of K . These are defined as follows [4]. For $0 < k < d$ the affine quermassintegrals $\Phi_0, \dots, \Phi_d$ of K are

$$\frac{1}{\Phi_k} = \frac{1}{\omega_d} \left(\int_{\Gamma} \left(\frac{\omega_{d-k}}{\text{Vol}_{d-k}(K|\Lambda)} \right)^d d\eta(\Lambda) \right)^{1/d},$$

while $\tilde{\Phi}_0 = \text{Vol}_d(K)$ and $\tilde{\Phi}_d = \omega_d$. The dual affine quermassintegrals $\tilde{\Phi}_0, \dots, \tilde{\Phi}_d$ of $K - \gamma$ are defined for $0 < k < d$ by

$$\tilde{\Phi}_{d-k}(K - \gamma) = \omega_d \left(\int_{\Gamma} \left(\frac{\text{Vol}_k(K \cap \Lambda^\perp)}{\omega_k} \right)^d d\eta(\Lambda) \right)^{1/d},$$

while $\tilde{\Phi}_0 = \text{Vol}_d(K)$ and $\tilde{\Phi}_d = \omega_d$.

Dividing both sides of (1) by $\text{Vol}_{d-k}(K|\Lambda)$, raising both sides to the power d , and integrating over all $\Lambda \in \Gamma$, we obtain

$$\text{Vol}_d(K)^d \int_{\Gamma} \frac{d\eta(\Lambda)}{\text{Vol}_{d-k}(K|\Lambda)^d} \leq \int_{\Gamma} \text{Vol}_k((K - \gamma) \cap \Lambda^\perp)^d d\eta(\Lambda),$$

which yields the inequality

$$\tilde{\Phi}_{d-k}(K - \gamma) \Phi_k(K) \geq \text{Vol}_d(K) \frac{\omega_d^2}{\omega_k \omega_{d-k}},$$

which is strict when $k \neq 0, d$.

REFERENCES

1. T. Bonnesen and W. Fenchel, *Theorie der Konvexen Körper*, Chelsea, New York, 1948.
2. Yu. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Springer-Verlag, New York, 1980.
3. G. D. Chakerian, *Inequalities for the difference body of a convex body*, Proc. Amer. Math. Soc. **18** (1967), 879–884.
4. E. Grinberg, *Isoperimetric inequalities and identities for k -dimensional cross-sections of convex bodies*, Math. Ann. **291** (1991), 75–86.
5. H. Guggenheimer and E. Lutwak, *A characterization of the n -dimensional parallelotope*, Amer. Math. Monthly **83** (1976), 475–478.
6. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag, Berlin and New York, 1957.
7. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, 1969.
8. E. Lutwak, *Mean dual and harmonic cross-sectional measures*, Ann. Mat. Pura Appl. **119** (1979), 139–148.
9. ———, *Intersection bodies and dual mixed volumes*, Adv. in Math. **71** (1988), 232–261.
10. R. T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, NJ, 1970.
11. C. A. Rogers and G. C. Shephard, *Convex bodies associated with a given convex body*, J. London Math. Soc. (2) **33** (1958), 270–281.

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GEORGIA 30332
E-mail address: spingarn@math.gatech.edu