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BOUNDEDNESS OF THE RIESZ POTENTIAL ON A COMPLETE MANIFOLD WITH NONNEGATIVE RICCI CURVATURE

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ABSTRACT. In this paper we obtain a necessary and sufficient condition for the boundedness of the Riesz potential on a complete manifold with nonnegative Ricci curvature.

In this paper, we consider the boundedness of the Riesz potential $(-\Delta)^{-\alpha/2}$ $(0<\alpha<n)$ on a complete Riemannian manifold with nonnegative Ricci curvature. The author [1] proved that: (1) if $V_x(r)\le C_n r^\beta$ $(\beta<n)$ then $(-\Delta)^{-\alpha/2}$ $(0<\alpha<n)$ is never of type (p, q) for any $p, q\ge 1$, and (2) if $V_x(r)\ge C_n r^n$ then $(-\Delta)^{-\alpha/2}$ $(0<\alpha<n)$ is of type (p, q) where $1<p<q<\infty$ if and only if $1/q=1/p-\alpha/n$. In this paper we improve this result and prove the following result.

Theorem. *Suppose M is a complete Riemannian manifold with nonnegative Ricci curvature. Then $(-\Delta)^{-\alpha/2}$ $(0<\alpha<n)$ is of type (p, q) $(1<p, q<\infty)$ if and only if $V_x(r)\ge C_n r^n$ for all $x\in M$ and $1/q=1/p-\alpha/n$. The condition is also necessary when $1\le p, q<\infty$.*

Proof. We only need to prove that the condition is necessary.

Assume that $(-\Delta)^{-\alpha/2}$ is of type (p, q) where $1\le p, q<\infty$. Then

$$\|(-\Delta)^{-\alpha/2}f\|_q\le C_{n,\alpha,p,q}\|f\|_p \tag{1}$$

for all $f\in L^p(M)$.

We set $f(y)=H(x, y, s)$ where $H(x, y, s)$ is the heat kernel of M, x is a fixed point in M, and s is a fixed positive constant.

$$\begin{aligned}(-\Delta)^{-\alpha/2}f(z)&=\Gamma\left(\frac{\alpha}{2}\right)^{-1}\int_0^\infty t^{\alpha/2-1}\int_M H(z, y, t)H(x, y, s)\,dy\,dt\\ &=\Gamma\left(\frac{\alpha}{2}\right)^{-1}\int_0^\infty t^{\alpha/2-1}H(x, z, t+s)\,dt.\end{aligned}\tag{2}$$

Using the estimate of the heat kernel [2]

$$H(x, y, t)\ge C_n\cdot\frac{1}{V_x(\sqrt{t})}e^{-\rho^2(x,y)/3t}, \tag{3}$$

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one has

$$(-\Delta)^{-\alpha/2} f(z) \geq \Gamma\left(\frac{\alpha}{2}\right)^{-1} C_n \int_0^\infty t_0^{\alpha/2-1} \frac{1}{V_x(\sqrt{t+s})} e^{-\rho^2(x,z)/3(s+t)}\, dt.$$

By the Bishop comparison theorem we have

$$\frac{V_x(\sqrt{s})}{V_x(\sqrt{s+t})} \geq \left(\frac{\sqrt{s}}{\sqrt{t+s}}\right)^n,$$

so

$$\begin{aligned}(-\Delta)^{-\alpha/2} f(z) &\geq C_{n,\alpha} \frac{1}{V_x(\sqrt{s})} \int_0^\infty t^{\alpha/2-1} \left(\frac{\sqrt{s}}{\sqrt{t+s}}\right)^n e^{-\rho^2(x,z)/3(t+s)}\, dt \\ &\geq C_{n,\alpha} \frac{s^{\alpha/2}}{V_x(\sqrt{s})} e^{-\rho^2(x,z)/3s},\end{aligned} \tag{4}$$

$$\|(-\Delta)^{-\alpha/2} f(z)\|_q \geq C_{n,\alpha,q} \frac{s^{\alpha/2}}{(V_x(\sqrt{s}))^{1-1/q}} \left(\frac{1}{V_x(\sqrt{s})} \int_M e^{-q\rho^2(x,z)/3s}\, dz\right)^{1/q}. \tag{5}$$

We choose the geodesic spherical coordinates about x in M. Since the measure of $\mathrm{Cut}(x) = 0$, we may ignore it and assume $dV = \sqrt{g(\rho,\theta)}\, d\theta\, d\rho$.

$$\begin{aligned}\frac{1}{V_x(\sqrt{s})} \int_M e^{-[q\cdot\rho^2(x,z)]/3s}\, dz &= \frac{1}{V_x(\sqrt{s})} \int_0^\infty e^{-(q\cdot\rho^2)/3s}\, dV_x(\rho) \\ &= \frac{1}{V_x(\sqrt{s})} \int_0^\infty V_x(\rho) e^{-(q\cdot\rho^2)/3s} \frac{2q\rho}{3s} \cdot d\rho.\end{aligned} \tag{6}$$

Using the Bishop comparison theorem we have

$$\frac{V_x(\sqrt{s})}{V_x(\rho)} \leq 1 + \left(\frac{\sqrt{s}}{\rho}\right)^n, \qquad \frac{V_x(\rho)}{V_x(s)} \geq \frac{1}{1+(\sqrt{s}/\rho)^n}. \tag{7}$$

Substituting (7) into (6) yields

$$\frac{1}{V_x(\sqrt{s})} \int_M e^{-[q\cdot\rho^2(x,z)]/3s}\, dz \geq \frac{2}{3} q \int_0^\infty \frac{1}{1+(\sqrt{s}/\rho)^n} \cdot e^{-(q\cdot\rho^2)/3s} \frac{\rho}{s}\, d\rho = C_{n,q}. \tag{8}$$

Substituting (8) into (5) yields

$$\|(-\Delta)^{-\alpha/2} f(z)\|_q \geq C_{n,\alpha,q} \frac{s^{\alpha/2}}{(V_x(\sqrt{s}))^{1-1/q}}. \tag{9}$$

On the other hand, clearly one obtains [2]

$$\|f(z)\|_p \leq C_{n,p} \frac{1}{(V_x(\sqrt{s}))^{1-1/p}}. \tag{10}$$

Substituting (9) and (10) into (1) we have

$$s^{\alpha/2}(V_x(\sqrt{s}))^{1/q-1} \leq C_{n,p,q,\alpha}(V_x(\sqrt{s}))^{1/p-1}. \tag{11}$$

So

$$(V_x(\sqrt{s}))^{1/p-1/q} \geq C_{n,p,q,\alpha} s^{\alpha/2}. \tag{12}$$

If $1/p - 1/q \leq 0$, inequality (12) could not hold for s sufficiently large; therefore, we may assume $1/p - 1/q > 0$. So

$$(\sqrt{s})^{n(1/p-1/q)} \geq C_{n,p,q,\alpha} s^{\alpha/2} \quad \text{for all } s > 0; \tag{13}$$
$$\text{so } 1/p - 1/q = \alpha/n, \quad \text{that is, } 1/q = 1/p - \alpha/n.$$

Substituting (13) into (12) yields

$$V_x(\sqrt{s}) \geq C_{n,p,q,\alpha}(\sqrt{s})^n \tag{14}$$

for all $x \in M$. Equations (13) and (14) imply that the condition is necessary.

References

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