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ON THE SHAPE OF THE UNIT SPHERE IN $Q(\Delta)$

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ABSTRACT. We show that the unit sphere in the Banach space of L^1 holomorphic quadratic differentials on the disk is weakly uniformly convex with exponent $1/2$ at certain points.

0. INTRODUCTION

Recall that a *holomorphic quadratic differential* ϕ on a Riemann surface R is an assignment of a holomorphic function $\phi_i(z_i)$ to each local coordinate z_i on R, subject to the transformation law

$$\phi_i(z_i) \cdot \frac{dz_i^2}{dz_j^2} = \phi_j(z_j)$$

on overlapping coordinate neighborhoods. It follows that the area element $|\phi(z)|\,|dz|^2$ is globally defined on R, and the L^1 *norm* of ϕ is

$$|\phi|_1 = \int_R |\phi(z)|\,|dz|^2.$$

With this norm, the L^1 holomorphic quadratic differentials on any Riemann surface R form a Banach space which we denote $Q(R)$, and the unit sphere in $Q(R)$ is denoted $S(R)$. When R is a Riemann surface of finite type, the space $Q(R)$ is the cotangent space to the *Teichmüller space* of marked conformal structures on R [Ga].

We will focus on the special case when R is equal to the open unit disk Δ. Since Δ is contractible, $Q(\Delta)$ can be identified with the Banach space $L_a^1(\Delta)$ of holomorphic L^1-functions on Δ. It is nevertheless advantageous to think of the elements of $Q(\Delta)$ as quadratic differentials. Specifically, there is an action on $Q(\Delta)$ by the Möbius group $\mathscr{M}$ of hyperbolic isometries of the disk defined by

$$M^*(\phi) = \phi \circ M \cdot (M')^2$$

which preserves the L^1 norm.

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Definition. A Banach space X is *smooth* if its unit sphere S is a smooth submanifold in the following weak sense: at every $x \in S$, the limit

$$\lim_{t \to 0} \frac{|x+ty| - |x|}{t}$$

exists for all directions $y \in S$. This limit is the *Gateaux derivative* of the norm at x in the direction y, and it is written $x_*(y)$. In fact, x_* extends to the unique linear functional on X for which $x_*(x) = 1$ [D].

Consider the (nonsymmetric) pairing $\langle \cdot , \cdot \rangle : S(\Delta) \times S(\Delta) \to \mathbf{R}$ defined by

$$\langle \phi , \psi \rangle = \operatorname{Re} \int_\Delta \phi(z) \frac{\overline{\psi(z)}}{|\psi(z)|} |dz|^2 .$$

A straightforward calculation shows that

$$\langle \phi , \psi \rangle = \phi_*(\psi)$$

is the Gateaux derivative of ϕ in the direction ψ. Note that for $M \in \mathscr{M}$,

$$\langle M^* \phi , M^* \psi \rangle = \langle \phi , \psi \rangle .$$

Translating into local coordinates, the quantity $\langle \phi , \psi \rangle$ is equal to the cosine of the difference $\arg(\phi) - \arg(\psi)$ averaged over R with respect to the area measure $|\phi(z)|\,|dz|^2$. Consequently, if ϕ is L^1 close to ψ then $\langle \phi , \psi \rangle$ is nearly 1. Indeed,

$$\begin{aligned} \varepsilon = |\psi - \phi|_1 &\geq \int_\Delta |\psi(z) - \phi(z)| \cos(\arg(\psi(z) - \phi(z)) - \arg(\psi(z)))\, |dz|^2 \\ &= \langle \psi - \phi , \psi \rangle = 1 - \langle \phi , \psi \rangle . \end{aligned}$$

The converse, however, does not hold.

Definition. A smooth Banach space X is *weakly uniformly convex* at a point x in the unit sphere S if, for all directions $y \in S$, $x_*(y) = 1 - \varepsilon$ implies that $|x - y|_1 = \delta$ where $\delta \to 0$ as $\varepsilon \to 0$. If δ can be chosen to be $O(\varepsilon^\alpha)$, then X is weakly uniformly convex at $x \in S$ with exponent α.

It is known that there are points in $S(\Delta)$ at which $Q(\Delta)$ is *not* uniformly convex. See [Mc, §5] or §3 of this article. By contrast McMullen proves that $Q(\Delta)$ is weakly uniformly convex at the constant differential $\psi(z) = (1/\pi) \cdot dz^2$ with exponent $\alpha < 1/6$. Here, we improve his estimate:

Theorem. *The Banach space* $Q(\Delta)$ *is weakly uniformly convex at* $\psi = 1/\pi \cdot dz^2$ *with exponent* $1/2$.

Definition. A Banach space X is *uniformly convex* at a point x in the unit sphere if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|y + x| = 2 - \varepsilon \Rightarrow |y - x| < \delta .$$

If δ can be chosen to be $O(\varepsilon^\alpha)$, then X is uniformly convex at x with exponent α.

A Hilbert space is uniformly convex with exponent $1/2$ at every point. By contrast, $Q(\Delta)$ is nowhere uniformly convex, as is illustrated by the following example due to McMullen.

Example. Fix $\psi \in S(\Delta)$, $\varepsilon > 0$, and choose $\phi \in S(\Delta)$ with the following property: there exist disjoint subsets $A, B \subset \Delta$ such that

$$\int_A |\phi|\,|dz|^2 > 1-\varepsilon \quad \text{and} \quad \int_B |\psi|\,|dz|^2 > 1-\varepsilon.$$

This can be achieved by taking a sequence M_n of Möbius transformations which tend to ∞ in $\mathscr{M}$ and setting $\phi = M_n^*\psi$ for n sufficiently large. Then

$$\begin{aligned}|\phi \pm \psi|_1 &\geq \int_A |\phi(z) \pm \psi(z)|\,|dz|^2 + \int_B |\phi(z) \pm \psi(z)|\,|dz|^2 \\ &> \int_A |\phi(z)|\,|dz|^2 - \int_A |\psi(z)|\,|dz|^2 + \int_B |\psi(z)|\,|dz|^2 - \int_B |\phi(z)|\,|dz|^2 \\ &= 2 - 4\varepsilon.\end{aligned}$$

This article is organized as follows: §1 recalls the construction of a particular bounded linear projection from $L^1(\Delta)$ to $L^1_a(\Delta)$. This projection is used in §2 to prove the theorem. The concluding §3 describes connections between $Q(\Delta)$ and the Bloch space and mentions several open problems.

1. A reproducing kernel

The material in this section is standard; references are [A, Ga, FR].

Recall that the density function of the Poincaré metric on the disk Δ is given by the formula

$$\rho(z) = 1/(1-|z|^2).$$

Let $K\colon \Delta \times \Delta \to \mathbf{C}$ be defined by

$$K(z, w) = 1/(1-z\overline{w})^4.$$

For each $z \in \Delta$,

$$\sup_{w\in\Delta} \rho(w)^{-2}|K(z, w)| < \infty.$$

Therefore,

$$Tf(z) = \frac{3}{\pi}\cdot\int_\Delta \rho(w)^{-2}K(z, w)f(w)\,|dw|^2$$

is a holomorphic function on the disk whenever $f \in L^1(\Delta)$.

For any w and any Möbius transformation M,

$$\int_\Delta |K(z, w)|\,|dz|^2 = |M'(w)|^2\cdot\int_\Delta |K(z, M(w))|\,|dz|^2.$$

Since $\int_\Delta |K(z, 0)|\,|dz|^2 = \pi$, it follows that $\int_\Delta |K(z, w)|\,|dz|^2 = \rho(w)^2$. This fact, together with Fubini's theorem, implies that $Tf \in L^1(\Delta)$ whenever f is

$$\begin{aligned}\int_\Delta |Tf(z)|\,|dz|^2 &= \frac{3}{\pi}\cdot\int_\Delta\left(\int_\Delta |\rho(w)^{-2}K(z, w)f(w)|\,|dz|^2\right)|dw|^2 \\ &\leq 3\cdot\int_\Delta |f(w)|\,|dw|^2.\end{aligned}$$

We will need the following

Proposition [FR]. *The transformation* $T\colon L^1(\Delta) \to L^1_a(\Delta)$ *is a bounded linear projection with norm at most* 3 *that maps antiholomorphic functions to constants. More precisely, when* $\phi \in L^1_a(\Delta)$,

(1) $T\phi = \phi$,

(2) $T\overline{\phi} = \overline{\phi(0)}$.

Proof. It remains to prove statements (1) and (2). Since holomorphic functions satisfy the mean value property, we have

$$\phi(0) = \frac{1}{2\pi} \cdot \int_{|z|=r} \phi(re^{2\pi i\theta})\, d\theta = \frac{3}{\pi} \cdot \int_\Delta \rho(w)^{-2} K(0, w)\phi(w)\, |dw|^2 = T\phi(0).$$

Let $A \in \mathscr{M}$ be a Möbius transformation that takes 0 to z and set

$$\Phi(z) = \frac{A'(z)^2 \phi(A(z))}{A'(0)^2}.$$

Then,

$$\phi(z) = \Phi(0) = \frac{3}{\pi} \cdot \int_\Delta \rho(w)^{-2} K(z, w)\phi(w)\, |dw|^2 = T\phi(z),$$

proving (1).

A second application of the mean value property (to antiholomorphic functions) yields

$$\begin{aligned} \overline{\phi(0)A'(0)}^2 &= \frac{3}{\pi} \cdot \int_\Delta \rho(w)^{-2} \overline{\phi(w)A'(w)}^2\, |dw|^2 \\ &= \overline{A'(0)}^2 \cdot \int_\Delta \rho(w)^{-2} K(z, w)\overline{\phi(w)}\, |dw|^2 \end{aligned}$$

since $\overline{A'(w)}^2 = \overline{A'(0)}^2 \cdot K(z, w)$. This shows (2). □

Corollary. *If* $\phi \in L^1_a(\Delta)$ *is normalized so that* $\phi(0) = 0$, *then*

$$\int_\Delta |\operatorname{Re}\phi(z)|\, |dz|^2 \le 6 \cdot \int_\Delta |\operatorname{Im}\phi(z)|\, |dz|^2.$$

Proof.

$$T(\operatorname{Im}\phi) = \frac{-i}{2} \cdot T(\phi - \overline{\phi}) = \frac{-i}{2} \cdot \phi.$$

Therefore, $|\operatorname{Re}\phi|_1 \le |\phi|_1 = 2 \cdot |T(\operatorname{Im}\phi)|_1 \le 6 \cdot |\operatorname{Im}\phi(z)|_1$. □

2. The exponent of weak uniform convexity is 1/2

Proof of Theorem. We are given $\psi = 1/\pi \cdot dz^2$ and ϕ in $S(\Delta)$ satisfying

$$\langle \psi, \phi \rangle = \int_\Delta \operatorname{Re}\phi(z)\, |dz|^2 = 1 - \varepsilon.$$

This mean value theorem yields $\operatorname{Re}\phi(0) = (1-\varepsilon)/\pi$. Since $\int_\Delta |\phi(z)|\, |dz|^2 = 1$, the Cauchy-Schwarz inequality implies

$$\begin{aligned} \int_\Delta |\operatorname{Im}\phi(z)|\, |dz|^2 &= \int_\Delta (|\phi(z)|^2 - |\operatorname{Re}\phi(z)|^2)^{1/2} |dz|^2 \\ &\le \left(\int_\Delta |\phi(z)| - |\operatorname{Re}\phi(z)|\, |dz|^2 \right)^{1/2} \cdot \left(\int_\Delta |\phi(z)| + |\operatorname{Re}\phi(z)|\, |dz|^2 \right)^{1/2} \\ &\le \varepsilon^{1/2} \cdot 2^{1/2}. \end{aligned}$$

Finally,

$$\begin{aligned}|\phi-\psi|_1 &= \int_\Delta \left|\frac{1}{\pi}-\phi(z)\right| |dz|^2 \\ &\le \int_\Delta \left|\frac{1}{\pi}-\operatorname{Re}\phi(0)\right| |dz|^2 + \int_\Delta |\operatorname{Re}\phi(0)-\operatorname{Re}\phi(z)|\,|dz|^2 + \int_\Delta |\operatorname{Im}\phi(z)|\,|dz|^2 \\ &\le \frac{\varepsilon}{\pi} + 6\cdot\int_\Delta |\operatorname{Im}\phi(0)-\operatorname{Im}\phi(z)|\,|dz|^2 + \varepsilon^{1/2}\cdot 2^{1/2} \\ &= O(\varepsilon^{1/2})\end{aligned}$$

where the second inequality depends on the Corollary in §1. This completes the proof of the theorem. □

Remark. The proof given above can be used to show that $Q(\Delta)$ is weakly uniformly convex with exponent $1/2$ at:

(1) all constant differentials $(e^{i\theta}/\pi)\cdot dz^2$, and
(2) all differentials $M^*\psi$ where $M \in \mathscr{M}$ and ψ is a constant differential.

3. Open questions

Below we list a few of the open questions that arise in this discussion.

Problem 1. Is $Q(\Delta)$ weakly uniformly convex at differentials $\phi = ((n+2)/2\pi)\cdot z^n\,dz^2$? at differentials $\phi \neq 0$ which have holomorphic nonzero extensions to a neighborhood of the closed disk?

Problem 2. Characterize the *flat* points in $S(\Delta)$ where $Q(\Delta)$ is not weakly uniformly convex.

For completeness, we include McMullen's construction of a flat point in $Q(\Delta)$ [Mc, §5].

Example. Select a sequence of points $z_n \in \Delta$ which are the centers of a family of disjoint hyperbolic disks whose radii tend to ∞. Let M_n be a sequence of Möbius transformations such that $M_n(z_n) = 0$. Let $\psi = 1/\pi\cdot dz^2$, $\psi_n = M_n^*\psi$, and

$$\phi = c\cdot\sum_n 2^{-n}\cdot M_n^*\psi$$

where the constant is chosen so that $\phi \in S(\Delta)$. Then $\langle\psi_n, \phi\rangle \to 1$; however, $|\psi_n - \phi|_1$ does *not* tend to 0.

Problem 3. Let R be a finite type Riemann surface. Discuss the weak uniform convexity of $Q(R)$.

Recall from §0 that, for every $\psi \in Q(\Delta)$, the value $\langle\phi, \psi\rangle$ is the *Gâteaux* or *weak* derivative of the L^1 norm, in the sense that it is equal to

$$\lim_{t\to 0}\frac{|\psi+t\phi|_1 - |\psi|_1}{t}.$$

A stronger concept of differentiability is the following:

Definition. The L^1 norm is *Fréchet differentiable* at $\psi \in S(\Delta)$ if the limit

$$\langle \phi, \psi \rangle = \lim_{t \to 0} \frac{|\psi + t\phi|_1 - |\psi|_1}{t}$$

exists *uniformly* for all directions $\phi \in S(\Delta)$.

Problem 4. Is the norm on $Q(\Delta)$ Fréchet differentiable at $\psi = 1/\pi \cdot dz^2$?

The differentiability of the L^1 norm on $Q(R)$ has been used to show that the *Teichmüller metric* of a finite type Riemann surface is C^1 [R].

Problem 5. Is the Teichmüller metric on universal Teichmüller space a C^1 metric?

Let $\mathscr{B}$ denote the *Bloch space* of holomorphic functions $\phi: \Delta \to \mathbf{C}$ whose Bloch norm

$$|\phi|_{\mathscr{B}} = \sup_{z \in \Delta} \rho(z)^{-2} |\phi(z)|$$

is finite. There is a pairing $\langle\langle \cdot, \cdot \rangle\rangle : L^1_a(\Delta) \times \mathscr{B} \to \mathbb{C}$ defined by

$$\langle\langle \phi, \beta \rangle\rangle = \lim_{t \to 1} \frac{1}{\pi} \cdot \int_{t\Delta} \phi(z) \overline{\beta(z)}\, |dz|^2.$$

For each $\beta \in \mathscr{B}$, the map $\beta_*: L^1_a(\Delta) \to \mathbb{C}$, defined by $\beta_*(\phi) = \langle\langle \phi, \beta \rangle\rangle$, is linear, and the transformation $\beta \mapsto \beta_*$ is an isomorphism from $\mathscr{B}$ onto the dual space $L^1_a(\Delta)^*$ [A].

There is a bounded operator P that projects the space $L^\infty(\Delta)$ onto $\mathscr{B}$, defined by

$$Pf(z) = \frac{1}{\pi} \cdot \int_\Delta f(w)(1 - \overline{w} z)^{-2} |dw|^2.$$

Then $P: L_\infty(\Delta) \to \mathscr{B}$ is analogous to $T: L^1(\Delta) \to L^1_a(\Delta)$ defined in §1; in particular, $P(\phi) = \phi$ if ϕ is holomorphic. Identifying $L^1_a(\Delta)$ with $Q(\Delta)$, we obtain a composite map from $S(\Delta)$ to $L^\infty(\Delta)$ to $\mathscr{B}$ given by

$$\psi \to \psi/|\psi| \to P(\psi/|\psi|).$$

This composition maps $S(\Delta)$ to the Bloch space.

Problem 6. Which Bloch maps (or equivalently, which elements of $L^1_a(\Delta)^*$) arise by this construction?

A straightforward calculation with Taylor series shows that

$$P\frac{\psi}{|\psi|_*}(\phi) = \left\langle\left\langle \phi, P\frac{\psi}{|\psi|} \right\rangle\right\rangle = \int_\Delta \phi(z) \frac{\overline{\psi(z)}}{|\psi(z)|} |dz|^2,$$

which is just our original pairing $\langle \phi, \psi \rangle$ before taking the real part. Thus, the issue in Problem 6 is to analyze the operator $P: L_\infty(\Delta) \to \mathscr{B}$.

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