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SOME DEFORMATIONS OF THE HOPF FOLIATION ARE ALSO KÄHLER

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ABSTRACT. Fix $\alpha = (\alpha_0, \dots, \alpha_n) \in \overset{\circ}{\mathbf{R}}^{n+1}$. The trajectories of the flow on $\mathbf{S}^{2n+1} \subset \mathbf{C}^{n+1}$ given by

$$\phi_t : (z_0, \dots, z_n) \mapsto (z_0 e^{i\alpha_0 t}, \dots, z_n e^{i\alpha_n t})$$

constitute the leaves of a $2n$-codimensional (nonsingular) foliation of $\mathbf{S}^{2n+1}$. We use (locally defined) branches of the logarithm to give this foliation a (global) transverse Kähler structure.

INTRODUCTION

Fix $\alpha = (\alpha_0, \dots, \alpha_n) \in \overset{\circ}{\mathbf{R}}^{n+1} \equiv \mathbf{R}^{n+1} - \{0\}$. The trajectories of the flow on $\mathbf{S}^{2n+1} \subset \mathbf{C}^{n+1}$ given by

$$\phi_t : (z_0, \dots, z_n) \mapsto (z_0 e^{i\alpha_0 t}, \dots, z_n e^{i\alpha_n t})$$

constitute the 1-dimensional leaves (the flow has no fixed points) of a codimension $2n$ foliation $\mathscr{F}_\alpha$ of $\mathbf{S}^{2n+1}$. Aside from the $n+1$ great circles (degenerate tori) determined by $|z_j| = 1$, the leaves are skew lines on the tori determined by $|z_0| = \rho_0, \dots, |z_n| = \rho_n$, where each ρ_j is constant (and of course $\rho_0^2 + \cdots + \rho_n^2 = 1$).

We use (locally defined) branches of the logarithm to give this foliation a transverse Kähler structure (cf. [D, NT, S]). The difficulty here is showing that this Kähler structure is globally well defined.

Except when $\alpha = (\alpha_0, \dots, \alpha_0)$ (the Hopf fibration), the winding lines on the nondegenerate tori are not geodesics. Thus $\mathscr{F}_\alpha$ is a nonharmonic Kähler foliation (cf. [NT, 5.3]). Knowing that arbitrarily near a fixed Hopf foliation, there exist $\mathscr{F}_\alpha$ parametrized by $\overset{\circ}{\mathbf{R}}^{n+1}$, the (real) dimension of the space of (transverse) Kähler foliations near the Hopf foliation is at least $n+1$. The (complex) dimension of the space of (transverse) holomorphic foliations near the Hopf foliation is $(n+1)^2 - 1$ [DK, 2.16].

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THE TRANSVERSE KÄHLER STRUCTURE

When $\alpha = (1, \ldots, 1)$, $\mathscr{F}_\alpha$ is the Hopf fibration $\mathbf{S}^{2n+1} \to \mathbf{CP}^n$ induced by the map

$$(z_0, \ldots, z_n) \mapsto [z_0, \ldots, z_n].$$

The generalization of this fibration map for other values of α is a "root map" that is *only locally defined* but may be designated by

$$(z_0, \ldots, z_n) \mapsto [z_0^{\alpha_1 \cdots \alpha_n}, \ldots, z_j^{\alpha_0 \cdots \hat{\alpha}_j \cdots \alpha_n}, \ldots, z_n^{\alpha_0 \cdots \alpha_{n-1}}],$$

so that (locally) the leaf $(z_0 e^{i\alpha_0 t}, \ldots, z_n e^{i\alpha_n t})$ gets mapped to a single point since

$$\frac{(z_j e^{i\alpha_j t})^{\alpha_0 \cdots \hat{\alpha}_j \cdots \alpha_n}}{(z_k e^{i\alpha_k t})^{\alpha_0 \cdots \hat{\alpha}_k \cdots \alpha_n}} = \frac{z_j^{\alpha_0 \cdots \hat{\alpha}_j \cdots \alpha_n}}{z_k^{\alpha_0 \cdots \hat{\alpha}_k \cdots \alpha_n}}, \qquad j, k \in \{0, \ldots, n\}.$$

To define transversely holomorphic coordinate charts (see, for example, [S, §3.3]), fix $(z_0, \ldots, z_n) \in \mathbf{S}^{2n+1}$ with $z_k \neq 0$ and choose a branch for the $(\alpha_0 \cdots \hat{\alpha}_j \cdots \alpha_n)$th power of each z_j. Then the "root map" is locally well defined and images of the leaves are points. Moreover, the image under the root map of a neighborhood U in its domain may be identified with a neighborhood $V \subset \mathbf{CP}^n$ that (locally) can be given holomorphic coordinates $V \hookrightarrow \mathbf{C}^n$, and these induce transversely holomorphic coordinates on U:

$$\begin{array}{ccccc} U & \longrightarrow & \mathbf{R} \times V & \longrightarrow & \mathbf{R} \times \mathbf{C}^n \\ & & \downarrow & & \downarrow \\ & & V & \longrightarrow & \mathbf{C}^n \end{array}$$

One can also pull back the Fubini-Study metric on $\mathbf{CP}^n$ along these root maps to obtain a globally well-defined transverse Hermitian metric compatible with this "canonical" transverse almost complex structure. In particular, the imaginary part of the Fubini-Study metric is a closed 2-form $\Omega_{\mathbf{CP}^n}$, so the pulled-back 2-form Ω_α is also closed. The structure induced by the (locally defined) root maps forces Ω_α to have rank $2n$ and to be smooth and closed. We will now see that Ω_α is globally well defined.

To be more concrete, with respect to local coordinates in a particular homogeneous coordinate system on $\mathbf{CP}^n$, e.g.,

$$\zeta_j = u_j / u_0, \qquad j = 1, \ldots, n,$$

the Kähler form from the Fubini-Study metric is given by

$$\Omega_{\mathbf{CP}^n}(\zeta) = \frac{i}{2} \frac{(1 + |\zeta|^2) \sum_{\mu=1}^n d\zeta_\mu \wedge d\overline{\zeta}_\mu - \sum_{\mu,\nu=1}^n \overline{\zeta}_\mu \zeta_\nu d\zeta_\mu \wedge d\overline{\zeta}_\nu}{(1 + |\zeta|^2)^2}$$

[W, p. 190]. Denote $\alpha_{\widehat{j}} = \alpha_0 \cdots \widehat{\alpha_j} \cdots \alpha_n$, a product of n real numbers that we will encounter repeatedly. $\Omega_{\mathbf{CP}^n}$ can be pulled back using

$$\zeta_j = \frac{z_j^{\alpha_{\widehat{j}}}}{z_0^{\alpha_{\widehat{j}}}}, \qquad j = 1, \ldots, n,$$

$$1+|\zeta|^2 = 1+\sum_{j=1}^{n}\frac{|z_j|^{2\alpha_{\widehat{j}}}}{|z_0|^{2\alpha_{\widehat{0}}}} = \frac{\sum_{j=0}^{n}|z_j|^{2\alpha_{\widehat{j}}}}{|z_0|^{2\alpha_{\widehat{0}}}},$$

and

$$d\zeta_j = \frac{z_j^{\alpha_{\widehat{j}}-1}}{z_0^{\alpha_{\widehat{0}}+1}}(\alpha_{\widehat{j}}z_0\,dz_j - \alpha_{\widehat{0}}z_j\,dz_0)$$

to obtain

$$\begin{aligned}\Omega_\alpha(z) = \frac{i}{2(1+|\zeta|^2)^2}\Bigg\{&(1+|\zeta|^2)\sum_{\mu=1}^{n}\frac{|z_\mu|^{2(\alpha_{\widehat{\mu}}-1)}}{|z_0|^{2(\alpha_{\widehat{0}}+1)}}\\
&\times\Big(\alpha_{\widehat{\mu}}^2|z_0|^2dz_\mu\wedge d\overline{z}_\mu - \alpha_{\widehat{\mu}}\alpha_{\widehat{0}}z_0\overline{z}_\mu dz_\mu\wedge d\overline{z}_0\\
&\qquad - \alpha_{\widehat{0}}\alpha_{\widehat{\mu}}z_\mu\overline{z}_0dz_0\wedge d\overline{z}_\mu + \alpha_{\widehat{0}}^{\;2}|z_\mu|^2dz_0\wedge d\overline{z}_0\Big)\\
&-\sum_{\mu,\nu=1}^{n}\frac{z_\nu\overline{z}_\mu|z_\mu|^{2(\alpha_{\widehat{\mu}}-1)}|z_\nu|^{2(\alpha_{\widehat{\nu}}-1)}}{|z_0|^{2(2\alpha_{\widehat{0}}+1)}}\\
&\times\Big(\alpha_{\widehat{\mu}}\alpha_{\widehat{\nu}}|z_0|^2dz_\mu\wedge d\overline{z}_\nu - \alpha_{\widehat{\mu}}\alpha_{\widehat{0}}z_0\overline{z}_\nu dz_\mu\wedge d\overline{z}_0\\
&\qquad - \alpha_{\widehat{0}}\alpha_{\widehat{\nu}}z_\mu\overline{z}_0dz_0\wedge d\overline{z}_\nu + \alpha_{\widehat{0}}^{\;2}z_\mu\overline{z}_\nu dz_0\wedge d\overline{z}_0\Big)\Bigg\}.\end{aligned}$$

This 2-form is globally well defined since the only roots that occur are roots of real functions. Similarly, the associated (transverse) tensor

$$h_\alpha = \left(\sum h_{\alpha\mu\nu}(z)\,dz_\mu\otimes d\overline{z}_\nu\right)(1+|z|^2)^{-2}$$

is globally well defined. It is obviously compatible with its own natural transverse complex structure. We thus have a transverse Kähler structure on $\mathscr{F}_\alpha$, the 1-dimensional foliation of $\mathbf{S}^{2n+1}$.

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