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EXISTENCE OF A NONTRIVIAL SOLUTION TO A STRONGLY INDEFINITE SEMILINEAR EQUATION

B. BUFFONI, L. JEANJEAN, AND C. A. STUART

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ABSTRACT. Under general hypotheses, we prove the existence of a nontrivial solution for the equation $Lu = N(u)$, where u belongs to a Hilbert space H, L is an invertible continuous selfadjoint operator, and N is superlinear. We are particularly interested in the case where L is strongly indefinite and N is not compact. We apply the result to the Choquard-Pekar equation

$$-\Delta u(x) + p(x)u(x) = u(x)\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\,dy, \qquad u \in H^1(\mathbb{R}^3),\ u \neq 0,$$

where $p \in L^\infty(\mathbb{R}^3)$ is a periodic function.

1. INTRODUCTION

Let H be a real Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$. We consider in H the equation

$$Lu = N(u), \qquad u \neq 0, \tag{1.1}$$

where $L\colon H \to H$ is an invertible continuous selfadjoint operator and N is a superlinear operator whose properties are given in the next section. We suppose that $\sigma(L) \cap \mathbb{R}_+ \neq \varnothing$. Denoting by H_+ and H_- the eigenspaces of L corresponding to $\sigma(L) \cap \mathbb{R}_+$ and $\sigma(L) \cap \mathbb{R}_-$, respectively, there exists $\delta > 0$ such that

$$\forall u \in H_+ : \langle Lu, u\rangle \geq \delta||u||^2 \tag{1.2}$$

and

$$\forall u \in H_- : \langle Lu, u\rangle \leq -\delta||u||^2. \tag{1.3}$$

We call L *strongly indefinite* if the dimensions of H_+ and H_- are infinite. The equation (1.1) has been considered in some recent papers [1, 5–8] (see also [3, 4, 9–11] for related problems). In this article we give a new existence theorem for (1.1). The proof given is simpler and the theorem can be applied to treat

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a larger class of nonlinearities. In particular, we can handle an equation of the form

$$(1.4)\qquad -\Delta u(x) + p(x)u(x) = u(x)\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\,dy\,, \qquad u \in H^1(\mathbb{R}^3)\,,\ u \neq 0$$

(the function $p \in L^\infty(\mathbb{R}^3)$ is periodic), which is referred to in the solid-state physics literature as a Choquard-Pekar equation. Note that the linear part of (1.4) is in general strongly indefinite. Due to the fact that the nonlinear part is not compact, the "linking theorems" of Benci-Rabinowitz and their generalizations, which played in [5–8] a crucial role in the discussion of (1.1), cannot be used. They require, indeed at a deep level, the compactness of the nonlinearity.

2. The general result

We suppose that there exists $\phi \in C^2(H, \mathbb{R})$ such that
(H1) $N = \nabla\phi$.
(H2) $\lim_{||u||\to 0}(\phi(u)/||u||^2) = 0$ and $\exists p > 2$, $\forall u \in H : \langle N(u), u\rangle \geq p\phi(u)$.
(H3) ϕ is convex and $\phi(u) = 0 \Rightarrow u = 0$.
(H4) $\exists K, C > 0$ such that $\forall u \in H:$ $||N(u)|| \leq K\langle N(u), u\rangle + C$.
(H5) If there exists a bounded sequence $\{u_n\} \subset H$ such that

$$Lu_n - N(u_n) \to 0 \quad \text{and} \quad \liminf_{n\to\infty}\langle N(u_n), u_n\rangle > 0,$$

then there exists $u \neq 0$ with $Lu = N(u)$.

It follows from (H2) that $\phi(0) = 0$, $N(0) = 0$, and the function

$$\phi(tu)/t^p, \qquad t > 0,$$

is increasing in t for all $u \in H$. Together with (H3), this implies that

$$\forall u \neq 0: \ \phi(u) > 0.$$

Finally, if N is compact then (H5) holds. Indeed, if $\{u_n\}$ satisfies the conditions of (H5), passing to a subsequence, we can suppose that $N(u_n) \to w$. Therefore $Lu_n \to w$ and $u_n \to L^{-1}w := u$. Clearly $Lu = N(u)$ and $u \neq 0$.

Theorem 2.1. *Under the hypotheses* (H1) *to* (H5), *there exists* $u \in H$ *such that* $Lu = N(u)$ *and* $u \neq 0$.

The idea of the proof is to find a nontrivial critical point of the function

$$J(u) = \tfrac{1}{2}\langle Lu, u\rangle - \phi(u).$$

The first step is a kind of Lyapunov-Schmidt reduction. Let P denote the orthogonal projection onto H_+.

Lemma 2.1. *There exists* $g \in C^1(H_+, H_-)$ *such that*

$$(2.1)\qquad \forall w \in H_-:\ w \neq g(v) \Rightarrow J(v+w) < J(v+g(v)).$$

Moreover,

$$(2.2)\qquad Lg(v) = (I-P)N(v+g(v)).$$

Proof. For a given $v \in H_+$, the function f_v defined on H_- by

$$f_v(w) = \tfrac{1}{2}\langle Lv, v\rangle + \tfrac{1}{2}\langle Lw, w\rangle - \phi(v+w)$$

is strictly concave and $\lim_{||w||\to\infty} f(w) = -\infty$. Hence there exists $g(v) \in H_-$ verifying (2.1). Since $g(v)$ is a critical point of f_v, it satisfies (2.2). Finally, the implicit function theorem shows that $g \in C^1(H_+, H_-)$. Indeed, $g(v)$ is the unique w satisfying $Lw - (I-P)N(v+w) = 0$. The derivative of the left member with respect to w is the operator defined on H_- by $dw \to Ldw - (I-P)N'(v+w)dw$. This operator is selfadjoint; indeed,

$$\begin{aligned}
&\langle Ldw - (I-P)N'(v+w)dw\,, dz\rangle \\
&\quad = \langle Ldw\,, dz\rangle - \langle N'(v+w)dw\,, dz\rangle \\
&\quad = \langle Ldz\,, wz\rangle - \langle N'(v+w)dz\,, dw\rangle \\
&\quad = \langle Ldz - (I-P)N'(v+w)dz\,, dw\rangle
\end{aligned}$$

for all $dw\,, dz \in H_-$. Since

$$\begin{aligned}
&\langle Ldw - (I-P)N'(v+w)dw\,, dw\rangle \\
&\quad = \langle Ldw - N'(v+w)dw\,, dw\rangle \\
&\quad \le \langle Ldw\,, dw\rangle \quad \text{by the convexity of } \phi \\
&\quad \le -\delta||dw||^2 \quad \text{by } (1.3)
\end{aligned}$$

for all $dw \in H_-$, we can conclude that its inverse exists and is bounded.

Let us introduce the function F defined on H_+ by $F(v) = J(v + g(v))$. The next lemma allows us to restrict our attention to the critical points of F.

Lemma 2.2. $\forall v \in H_+ :\ ||\nabla F(v)|| = ||\nabla J(v + g(v))||$.

Proof. The relation (2.2) is equivalent to $(I-P)\nabla J(v+g(v)) = 0$. Hence we have for $dv \in H_+$:

$$\begin{aligned}
\langle \nabla F(v)\,, dv\rangle &= \langle \nabla J(v+g(v))\,, dv\rangle + \langle \nabla J(v+g(v))\,, g'(v)dv\rangle \\
&= \langle \nabla J(v+g(v))\,, dv\rangle
\end{aligned}$$

and so $||\nabla F(v)|| \le ||\nabla J(v+g(v))||$. Conversely, for $du \in H$,

$$\begin{aligned}
&\langle \nabla J(v+g(v))\,, du\rangle \\
&\quad = \langle \nabla J(v+g(v))\,, Pdu\rangle + \langle \nabla J(v+g(v))\,, (I-P)du\rangle \\
&\quad = \langle \nabla F(v)\,, Pdu\rangle
\end{aligned}$$

and so $||\nabla F(v)|| \ge ||\nabla J(v+g(v))||$.

Proof of the theorem. The main ingredient in the proof is a version of the well-known mountain pass lemma in which the Palais-Smale condition is not assumed [2, p. 943]. In order to use this result on F we shall prove that

(1) $F(0) = 0$;

(2) there exist $r, \alpha > 0$ such that

$$\forall v \in H_+ :\ ||v|| \le r \Rightarrow F(v) \ge 0 \text{ and } ||v|| = r \Rightarrow F(v) \ge \alpha;$$

(3) there exists $v \in H_+$ such that $F(v) < 0$.

The first point is a direct consequence of $g(0) = 0$. The first part of (H2) and (1.2) imply the existence of $r, \alpha > 0$ such that for all $v \in H_+$:

$$||v|| \le r \Rightarrow \tfrac{1}{2}\langle Lv\,, v\rangle - \phi(v) \ge 0$$

and

$$||v|| = r \Rightarrow \tfrac{1}{2}\langle Lv\,, v\rangle - \phi(v) \ge \alpha.$$

Now (2.1) implies $F(v) \geq J(v)$, which proves the second point. Fix $v \in H_+$ with $\|v\| = 1$. There exists $t > 0$ such that $F(tv) < 0$ (and so the third point holds). Indeed, consider a sequence $\{t_n\}$ such that $t_n \to \infty$. Setting $u_n = t_n v + g(t_n v)$ and $w_n = u_n/\|u_n\|$, we have $\|w_n\| = 1$ and so, passing to a subsequence, $w_n \rightharpoonup w = w_+ + w_-$, where $w_\pm \in H_\pm$. The subsequence can be chosen so that one of the following cases occurs: $\|g(t_n v)\|/t_n \to +\infty$ or $\|g(t_n v)\|/t_n \to \lambda \geq 0$. In the first case, for n sufficiently large, we get

$$\begin{aligned} 2F(t_n v) &= 2J(t_n v + g(t_n v)) \\ &\leq t_n^2 \langle Lv, v\rangle - \delta \|g(t_n v)\|^2 \quad \text{by (1.3)} \\ &< 0. \end{aligned}$$

In the second case, $t_n/\|u_n\| \to \mu = (1+\lambda^2)^{-1/2} > 0$ and so $w = \mu v + w_- \neq 0$. Hence

$$0 < \phi(w) \leq \liminf_{n\to\infty} \phi(w_n),$$

and there exists n_0 such that $\phi(w_n) > \frac{1}{2}\phi(w)$ for all $n \geq n_0$. Also

$$\begin{aligned} \phi(u_n) &= \phi(\|u_n\| w_n) \geq \|u_n\|^p \phi(w_n) \qquad \text{by (H2)} \\ &\geq \tfrac{1}{2} t_n^p \phi(w) \end{aligned}$$

for $n \geq n_0$ since we may assume $\|u_n\| \geq t_n \geq 1$. Thus for $n \geq n_0$,

$$2F(t_n v) = 2J(u_n) \leq t_n^2 \langle Lv, v\rangle - t_n^p \phi(w)$$

and so $F(t_n v) < 0$ for n sufficiently large.

Applying the mountain pass lemma, we see that there exists a sequence $\{v_n\} \subset H_+$ such that $\lim_{n\to\infty} F(v_n) = c > 0$ and $\nabla F(v_n) \to 0$. Setting $u_n = v_n + g(v_n)$, we have $\lim_{n\to\infty} J(u_n) = c > 0$ and $\nabla J(u_n) \to 0$ by definition of F and Lemma 2.2. Taking a subsequence, we can assume that $|J(u_n) - c| \leq 1/n$ and $\|\nabla J(u_n)\| \leq 1/n$. Now

$$\begin{aligned} \langle N(u_n), u_n\rangle &= \langle Lu_n, u_n\rangle - \langle \nabla J(u_n), u_n\rangle \\ &= 2J(u_n) + 2\phi(u_n) - \langle \nabla J(u_n), u_n\rangle \\ &\leq 2\left(c + \frac{1}{n}\right) + \frac{2}{p}\langle N(u_n), u_n\rangle + \frac{\|u_n\|}{n}; \end{aligned}$$

therefore

$$\langle N(u_n), u_n\rangle \leq \frac{p}{p-2}\left(2c + \frac{2}{n} + \frac{\|u_n\|}{n}\right),$$

and so

$$\begin{aligned} \delta\|u_n\| \leq \|Lu_n\| &\leq \frac{1}{n} + \|N(u_n)\| \\ &\leq \frac{1}{n} + C + K\langle N(u_n), u_n\rangle \quad \text{by (H4)} \\ &\leq \frac{1}{n} + C + K\frac{p}{p-2}\left(2c + \frac{2}{n} + \frac{\|u_n\|}{n}\right). \end{aligned}$$

This implies that $\{u_n\}$ is a bounded sequence. Finally

$$\begin{aligned} \langle N(u_n), u_n\rangle &= 2J(u_n) + 2\phi(u_n) - \langle \nabla J(u_n), u_n\rangle \\ &\geq 2\left(c - \frac{1}{n}\right) - \frac{\|u_n\|}{n} \end{aligned}$$

and therefore $\liminf_{n\to\infty}\langle N(u_n), u_n\rangle \geq 2c > 0$. The conclusion now follows from (H5).

3. Example

We shall now apply the general theory developed in §2 to prove the existence of a weak solution for equation (1.4). The Hilbert space H considered here is the Sobolev space $H^1(\mathbb{R}^3)$. The operator

$$S: D(S) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3),$$

defined by

$$D(S) = H^2(\mathbb{R}^3) \quad \text{and} \quad Su = -\Delta u + pu$$

with $p \in L^\infty(\mathbb{R}^3)$ periodic, is selfadjoint. Its spectrum is purely continuous, bounded from below and $\sigma(S) \cap \mathbb{R}_+ \neq \varnothing$. We suppose that p is such that $0 \notin \sigma(S)$. We associate with S the operator L defined on H by

$$\langle Lu, v\rangle = \int_{\mathbb{R}^3} \{\nabla u \nabla v + puv\}\, dx, \quad \forall u, v \in H.$$

L is selfadjoint, continuous, invertible and $\sigma(L)\cap\mathbb{R}_+ \neq \varnothing$ (the relation between S and L is discussed in [3]).

In order to study the nonlinear part, we need the following remark. For $u \in L^{4/3}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $u \geq 0$, set

$$f(x) = \int_{\mathbb{R}^3} \frac{u(y)}{|x-y|}\, dy.$$

We have

$$\begin{aligned} f(x) &= \int_{|x-y|\leq 1} \frac{u(y)}{|x-y|}\, dy + \int_{|x-y|\geq 1} \frac{u(y)}{|x-y|}\, dy \\ &\leq \|u\|_{L^2(\mathbb{R}^3)} \left\{\int_{|x-y|\leq 1} \frac{1}{|x-y|^2}\, dy\right\}^{1/2} \\ &\quad + \|u\|_{L^{4/3}(\mathbb{R}^3)} \left\{\int_{|x-y|\geq 1} \frac{1}{|x-y|^4}\, dy\right\}^{1/4} \\ &\leq C \max\{\|u\|_{L^2(\mathbb{R}^3)}, \|u\|_{L^{4/3}(\mathbb{R}^3)}\}. \end{aligned} \tag{3.1}$$

Hence for all $u, v \in H$, we obtain

$$\int_{\mathbb{R}^3} \left\{u(x) \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\, dy\right\} v(x)\, dx$$

$$\leq \|uv\|_{L^1(\mathbb{R}^3)} C \max\{\|u\|^2_{L^4(\mathbb{R}^3)}, \|u\|^2_{L^{8/3}(\mathbb{R}^3)}\} \tag{3.2}$$

$$\leq C\|u\|^3\|v\|, \tag{3.3}$$

and therefore there exists $N(u) \in H$ such that

$$\langle N(u), v\rangle = \int_{\mathbb{R}^3} \left\{u(x) \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\, dy\right\} v(x)\, dx.$$

The problem (1.4) consists now in finding $u \in H$, $u \neq 0$, such that $Lu = N(u)$. In this aim, we define for $u \in H$

$$\phi(u) = \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy.$$

It remains to check the hypotheses (H1) to (H5). Thanks to (3.1), we have

$$\phi(u) \leq ||u||^2_{L^2(\mathbb{R}^3)} C \max\{||u||^2_{L^4(\mathbb{R}^3)}, ||u||^2_{L^{8/3}(\mathbb{R}^3)}\} \leq C||u||^4 < \infty.$$

Moreover, $\phi \in C^2(H, \mathbb{R})$, $\nabla\phi = N$, and

$$\langle N'(u)v, z\rangle = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(y)v(x)z(x) + 2u(y)v(y)u(x)z(x)}{|x-y|}\,dx\,dy.$$

The hypothesis (H2) is also easy to check.

Lemma 3.1. *We have for all* $u, v \in H$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)v(x)u(y)v(y)}{|x-y|}\,dx\,dy \geq 0$$

and

$$\left\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)v^2(y)}{|x-y|}\,dx\,dy\right\}^2$$
$$\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\,dx\,dy.$$

Proof. Consider the bilinear form a on $C_0^\infty(\mathbb{R}^3)$ that is defined by

$$a(z, w) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z(x)w(y)}{|x-y|}\,dx\,dy.$$

Clearly a is symmetric. For $w \in C_0^\infty(\mathbb{R}^3)$ let

$$g(x) = \int_{\mathbb{R}^3} \frac{w(y)}{|x-y|}\,dy$$

be the corresponding Newtonian potential. Then $g \in C^2(\mathbb{R}^3)$ and

$$-\Delta g(x) = 4\pi w(x) \quad \forall x \in \mathbb{R}^3$$

and there exists $A > 0$ such that

$$|g(x)| \leq A|x|^{-1} \quad \text{and} \quad |\nabla g(x)| \leq A|x|^{-2} \qquad \forall |x| \geq 1.$$

Hence,

$$a(w, w) = \int_{\mathbb{R}^3} w(x)g(x)\,dx$$
$$= -\frac{1}{4\pi}\int_{\mathbb{R}^3} g(x)\Delta g(x)\,dx = \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla g(x)|^2\,dx \geq 0.$$

Also since a is positive definite and symmetric on $C_0^\infty(\mathbb{R}^3)$, it follows that

$$|a(w, z)| \leq a(w, w)^{1/2} a(z, z)^{1/2}.$$

It is easy to conclude, using the density of $C_0^\infty(\mathbb{R}^3)$ in $H^1(\mathbb{R}^3)$ and (3.1).

We deduce from Lemma 3.1 that $\langle N'(u)v, v\rangle \geq 0$ for all $u, v \in H$. Consequently ϕ is convex. Moreover $\phi(0) = 0 \Rightarrow u = 0$ and (H3) is verified. For $u, v \in H$, we have

$$\begin{aligned}
|\langle N(u), v\rangle| &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(y)u(x)v(x)}{|x-y|}\,dx\,dy\right| \\
&\leq \left\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy\right\}^{1/2} \left\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)u^2(y)}{|x-y|}\,dx\,dy\right\}^{1/2} \\
&\leq \langle N(u), u\rangle^{1/2}\langle N(u), u\rangle^{1/4}\langle N(v), v\rangle^{1/4} \quad \text{by Lemma 3.1} \\
&= \langle N(u), u\rangle^{3/4}\langle N(v), v\rangle^{1/4} \\
&\leq C\langle N(u), u\rangle^{3/4}\|v\| \quad \text{by (3.3)}.
\end{aligned}$$

Hence

$$\|N(u)\| \leq C\langle N(u), u\rangle^{3/4} \leq C\langle N(u), u\rangle + C$$

and (H4) is proved. Finally, let us consider a bounded sequence $\{u_n\} \subset H$ such that $Lu_n - N(u_n) \to 0$ and $\liminf_{n\to\infty}\langle N(u_n), u_n\rangle > 0$. If

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{y+B_R} u_n^2 dx = 0, \quad \forall R > 0$$

(B_R denotes the open ball of radius R centered at the origin), then by a result by Lions [13, Lemma I.1] $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for all $q \in]2, 6[$ and, therefore, $\langle N(u_n), u_n\rangle \to 0$ by (3.2), which is a contradiction; otherwise, passing to a subsequence, we get

$$\exists R < \infty, \quad \{y_n\} \subset \mathbb{R}^3 \text{ s.t. } \liminf_{n\to\infty} \int_{y_n+B_R} u_n^2\,dx > 0.$$

Using the periodicity of p and translating each u_n, we can find $R < \infty$ and a sequence $\{\tilde{u}_n\}$ such that

$$L\tilde{u}_n - N(\tilde{u}_n) \to 0 \quad \text{and} \quad \liminf_{n\to\infty} \int_{B_R} \tilde{u}_n^2\,dx > 0.$$

Passing to a subsequence, we have $\tilde{u}_n \rightharpoonup u$, and the compact inclusion $H^1(B_R) \subset L^2(B_R)$ shows that $u \neq 0$. Since $N(\tilde{u}_n) \rightharpoonup N(u)$, taking the limit in $L\tilde{u}_n - N(\tilde{u}_n) \to 0$, we obtain $Lu = N(u)$. To establish that $N(\tilde{u}_n) \rightharpoonup N(u)$, set

$$f_n(x) = \int_{\mathbb{R}^3} \frac{\tilde{u}_n^2(y)}{|x-y|}\,dy$$

and define f similarly with $\tilde{u}_n$ replaced by u. For $v \in H$,

$$|\langle N(\tilde{u}_n) - N(u), v\rangle| \leq \left|\int_{\mathbb{R}^3} (\tilde{u}_n - u) f v\,dx\right| + \|\tilde{u}_n\|\,\|(f_n - f)v\|_{L^2(\mathbb{R}^3)}.$$

Since $fv \in L^2(\mathbb{R}^3)$, we need only prove that $\lim_{n\to\infty}\|(f_n - f)v\|_{L^2(\mathbb{R}^3)} = 0$. By estimates like (3.1), there exists $K > 0$ such that $|f_n(x)| \leq K$ $\forall x \in \mathbb{R}^3$ and $\forall n \in \mathbb{N}$. Also for all $R > 0$

$$\begin{aligned}
|f_n(x) - f(x)| &\leq (4\pi R)^{1/2}\|\tilde{u}_n^2 - u^2\|_{L^2(B(x,R))} + \left(\frac{4\pi}{R}\right)^{1/4}\|\tilde{u}_n^2 - u^2\|_{L^{4/3}(\mathbb{R}^3)} \\
&\leq K\left\{R^{1/2}\|\tilde{u}_n - u\|_{L^4(B(x,R))}\|\tilde{u}_n + u\| + R^{-1/4}(\|\tilde{u}_n\|^2 + \|u\|^2)\right\}
\end{aligned}$$

$\forall x \in \mathbb{R}^3$ and $\forall n \in \mathbb{N}$. Since the inclusion $H^1(B(x, R)) \subset L^4(B(x, R))$ is compact, this shows that the subsequence $\tilde{u}_n$ can be chosen so that $f_n(x) \to f(x)$, $\forall x \in \mathbb{R}^3$. Then by dominated convergence we can conclude that

$$\lim_{n\to\infty} ||(f_n - f)v||_{L^2(\mathbb{R}^3)} = 0.$$

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DÉPARTEMENT DE MATHÉMATIQUES, ECOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, 1015 LAUSANNE, SWITZERLAND

E-mail address: stuart@masg3.epfl.ch